UNIVERSITY OF CALIFORNIA, SAN DIEGO

Isosymmetric Linear Transformations on Complex Hilbert Space

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in Mathematics

by

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The dissertation of Mark Stankus is approved, and it is acceptable in quality and form for publication on microfilm:

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PUBLICATIONS

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FIELDS OF STUDY

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Abstract of the Dissertation

Isosymmetric Linear Transformations on Complex Hilbert Space

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Professor Jim Agler, Chair

We explore the elementary operator theory of the equation

$$\sum_{m,n=1}^{\infty} c_{m,n} T^{*n} T^{m} = 0 \tag{0.1}$$

for $c_{m,n} \in \mathbf{C}$, $c_{m,n}$ nonzero for only finitely many m, n and T a bounded linear transformation on a complex Hilbert space in Chapters 1 and 2. We explore the equation

$$T^{*2}T - T^{*}T^{2} + T - T^{*} = 0 \tag{0.2}$$

in greater depth in Chapters 4 and 5.

Chapter 1 explores the algebraic and C^*-algebraic aspects of the equation (0.1) and both the spectral picture of and growth conditions on the resolvent of the operator T satisfying (0.1).

Chapter 2 explores the implications of Rosenblum's Theorem to the study of (0.1). These implications are sufficient in some cases to completely classify a solution to (0.1) given information about the spectrum of T. Chapter 2 also recalls a few definitions and results from the theory of von Neumann algebras which will be used in the rest of the paper.

Chapter 3 guarantees the existence of maximal invariant subspaces $\mathcal{M}$ for an operator T such that T restricted to $\mathcal{M}$ is a member of a fixed family of

operators. This provides an approach to completely solving the equation (0.1) for T for certain choices of $c_{m,n}$.

In Chapters 4 and 5, we study operators T satisfying (0.2). These operators are termed *isosymmetries.* The results of Chapters 1, 2 and 3 *do not* solve equation (0.2).

Chapter 4 gives the elementary operator theory of isosymmetries.

Chapter 5 classifies several collections of isosymmetries. Indeed, if T is an isosymmetry and T is hyponormal, T is a contraction, $Im(T) \geq 0$ or $Im(T) \leq 0$, then T is subnormal and the minimal normal extension of T has the same properties. If $T^*T \geq 1$, then T is the restriction to an invariant subspace of a direct integral of rank one perturbations of the unilateral shift. If the spectrum of T equals its boundry, then T has the form of a direct integral of 1×1 and 2×2 matrices. These constraints arise naturally from the analysis of Chapter 4.

Chapter 1

The Hereditary Functional Calculus, The Spectrum and Resolvent Inequalities

In this chapter, we introduce the notation of the paper, describe the Hereditary Functional Calculus [Ag1] for elements of a C^*-algebra and give the elementary operator theory for roots of a hereditary polynomial.

Within this paper, all Hilbert spaces will be complex and separable. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, we let $\mathcal{L}(\mathcal{H},\mathcal{K})$ denote the bounded linear transformations from $\mathcal{H}$ to $\mathcal{K}$. We denote $\mathcal{L}(\mathcal{H},\mathcal{H})$ by $\mathcal{L}(\mathcal{H})$. If $p \in \mathbf{C}[x,y]$, then, for $m \geq 0$ and $n \geq 0$, $p^\wedge(m,n)$ will be complex numbers such that $p(x,y) = \sum_{m,n} p^\wedge(m,n) y^n x^m$. Note that $p^\wedge(m,n)$ is zero for all but finitely many m and n. Unless otherwise indicated, whenever m and n are indices of summation, m and n will range over the positive integers. Throughout this paper, whenever a summation is taken over an infinite index set (e.g., $\sum_{i \in I} a_i$), all but finitely many summands are zero ($\{i \in I : a_i \neq 0\}$ is finite). This fact will be used tacitly many times in this paper. If m and n are positive integers and R is a ring, let $R^{m,n}$ denote m by n matrices with entries in R.

If C is a unital C^*-algebra, $c \in C$ and $p \in \mathbf{C}[x,y]$, then $p(c) \in C$ is defined by

$$p(c) = \sum_{m,n} p^\wedge(m,n)c^{*n}c^m. \tag{1.1}$$

If n and K are positive integers, $M_k \in \mathbf{C}^{n,n}$ and $p_k \in \mathbf{C}[x,y]$ for every $1 \le k \le K$ and $p = \sum_{k=1}^{K} M_k \otimes p_k \in \mathbf{C}^{n,n} \otimes \mathbf{C}[x,y]$, then $p(c) \in \mathbf{C}^{n,n} \otimes C$ is defined by

$$p(c) = \sum_{k=1}^{K} M_k \otimes p_k(c)$$

where $p_k(c)$ is defined as in (1.1). This functional calculus is termed the *Hereditary Functional Calculus* in [Ag1] and has been used to solve a number of problems [Ag1,Ag2, Ag3, Ag4, Ag5, Ag6, Ag7, Ag-S, C-P, M1, M2, H1].

In the following proposition, we record some elementary algebraic facts about the Hereditary Functional Calculus which are given in [Ag1, Ag6, Ag-S]. For $p \in \mathbf{C}[x,y]$, let $p^\vee$ be the unique member of $\mathbf{C}[x,y]$ which satisfies the relation

$$p^\vee(\lambda, \overline{\mu}) = \overline{p(\overline{\mu}, \lambda)} \qquad \text{for all} \qquad \lambda, \mu \in \mathbf{C}\,.$$

Explicitly, $p^\vee(x,y) = \sum_{m,n} \overline{p^\wedge(n,m)}y^n x^m$.

Proposition 1.2. *Let C be a C^*-algebra and $c \in C$. The map $p \mapsto p(c)$ from $\mathbf{C}[x,y]$ into C is a vector space homomorphism and the following hold.*

(a) $p \mapsto p(c)$ is an algebra homomorphism if and only if c is normal.

(b) For $p \in \mathbf{C}[x,y]$, $p(c)^* = p^\vee(c)$.

(c) If $p \in \mathbf{C}[x]$, $q \in \mathbf{C}[x,y]$, $s \in \mathbf{C}[y]$, and $t(x,y) = s(y)q(x,y)p(x)$, then

$$t(c) = s(c^*)q(c)p(c)\ , \tag{1.3}$$

where $p(c)$ and $s(c^*)$ are defined as in the Riesz Functional Calculus.

(d) If $p \in \mathbf{C}[x]$, then the definition of $p(c)$ from the Hereditary Functional Calculus

and the Riesz Functional Calculus agree.

*(e) If D is a unital C^*-algebra and $\pi : C \to D$ is a unital *-representation, then*

$$\pi(p(c)) = p(\pi(c)). \tag{1.4}$$

In the following lemma, several spatial properties of the Hereditary Functional Calculus are given for the special case when the C^*-algebra is $\mathcal{L}(\mathcal{H})$. The lemma is proved implicitly in [Ag1].

Proposition 1.5. *Let $p \in \mathbf{C}[x,y]$. If I is an index set, $T_\alpha \in \mathcal{L}(\mathcal{H}_\alpha)$ for each $\alpha \in I$ and $\sup\{\|T_\alpha\| : \alpha \in I\} < \infty$, then*

$$p\left(\bigoplus_{\alpha\in I} T_\alpha\right) = \bigoplus_{\alpha\in I} p(T_\alpha)\,. \tag{1.6}$$

*If $T \in \mathcal{L}(\mathcal{H})$ and $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is a unital *-representation, then*

$$\pi(p(T)) = p(\pi(T))\,. \tag{1.7}$$

If $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{M}$ is an invariant subspace for T, $P_{\mathcal{M}} \in \mathcal{L}(\mathcal{H})$, and $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$, then

$$p(T|\mathcal{M}) = P_{\mathcal{M}}p(T)|\mathcal{M}\,. \tag{1.8}$$

The rest of Chapters 1, 2 and 3 will be related to the following definition.

Definition 1.9. *Let $p \in \mathbf{C}[x,y]$ and c be a member of some unital C^*-algebra. c is a root of p if $p(c) = 0$ where $p(c)$ is defined by (1.1).*

Before continuing, we note that Propositions 1.2 and 1.5 have specific implications concerning roots.

Proposition 1.2 (e) implies that the property of being a root is closed with respect to unital *-representations. That is, let C be a unital C^*-algebra, $c \in C$,

$p \in \mathbf{C}[x,y]$, D be a unital C^*-algebra and $\pi : C \to D$ be a unital *-representation. If c is a root of p, then $\pi(c)$ is a root of p.

Proposition 1.5 implies that the property of being an operator and a root is closed with respect to direct sums (by (1.6)), unital *-representations (by (1.7)) and restrictions to invariant subspaces (by (1.8)). If $U \in \mathcal{L}(\mathcal{H})$ is unitary, then the map $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, defined by $\pi(X) = U^*XU$ for $X \in \mathcal{L}(\mathcal{H})$, is a unital *-representation. Therefore, any operator unitarily equivalent to T is a root of p if $T \in \mathcal{L}(\mathcal{H})$, $p \in \mathbf{C}[x,y]$ and T is a root of p.

By Proposition 1.2 (d), we see that T is *algebraic* if it is the root of some nonzero $p \in \mathbf{C}[x]$. We will explore roots of hereditary polynomials $p \in \mathbf{C}[x,y]$ from the point of view of generalizing certain results about algebraic operators. Note that the definition of root is broad enough to include, for example, the study of *isometries* (i.e., $(yx-1)(T)=0$) and *self-adjoint* operators (i.e., $(y-x)(T)=0$).

If $T \in \mathcal{L}(\mathcal{H})$, then $\{p \in \mathbf{C}[x] : p(T) = 0\}$ is an ideal of $\mathbf{C}[x]$. The following proposition of Jim Agler gives the analogous result in the hereditary situation.

Proposition 1.10. *Let C be a C^*-algebra and $c \in C$. The set*

$$\{p \in \mathbf{C}[x,y] : p(c) = 0\} \tag{1.11}$$

is an ideal of $\mathbf{C}[x,y]$. In addition, if $T \in \mathcal{L}(\mathcal{H})$, I is an ideal of $\mathbf{C}[x,y]$ such that $p(T) = 0$ for every $p \in I$, $p_1 \in \mathbf{C}[x,y]$, $p_2 \in \mathbf{C}[x,y]$ and $p_1 + I = p_2 + I$, then $p_1(T) = p_2(T)$.

Proof. The second statement follows trivially from Proposition 1.2. Let I be the set (1.11). By Proposition 1.2, I is a vector subspace of $\mathbf{C}[x,y]$. If $p, q \in I$, then, by Proposition 1.2 and (1.3),

$$(pq)(c) \;=\; \left(\sum_{m,n} q^\wedge(m,n) y^n p(x,y) x^m\right)(c)$$

$$\begin{aligned} &= \sum_{m,n} q^\wedge(m,n)\,(y^n p(x,y) x^m)\,(c) \\ &= \sum_{m,n} q^\wedge(m,n) c^{*^n} p(c) c^m \\ &= 0\,. \end{aligned}$$

Therefore, I is an ideal of $\mathbf{C}[x,y]$ and the proof of Proposition 1.10 is complete.

If T is an $n \times n$ matrix, then the spectrum of T is a subset of the (scalar) roots of p whenever $p \in \mathbf{C}[x]$ and $p(T) = 0$. The following proposition proves that a corresponding phenomena holds for the approximate point spectrum and the (scalar) roots of p whenever $p \in \mathbf{C}[x,y]$ and $p(T) = 0$. If $T \in \mathcal{L}(\mathcal{H})$, then $W(T)$ will denote the *numerical range* of T, $\sigma_p(T)$ will denote the set of eigenvalues (or *point spectrum*) of T, $\sigma_{ap}(T)$ will denote the approximate point spectrum of T and T will be called *finitely cyclic* if there exists finitely many vectors $\gamma_1, \ldots, \gamma_n$ such that the smallest invariant subspace for T containing $\gamma_1, \ldots, \gamma_n$ is $\mathcal{H}$.

Proposition 1.12. *Let $T \in \mathcal{L}(\mathcal{H})$ and $p \in \mathbf{C}[x,y]$. The approximate point spectrum of T is a subset of*

$$\{\lambda \in \mathbf{C} : p(\lambda, \overline{\lambda}) \in W(p(T))^-\}.$$

In particular, if $p(T) = 0$ and $\lambda \in \sigma_{ap}(T)$, then $p(\lambda, \overline{\lambda}) = 0$. If T is finitely cyclic, then the essential spectrum of T is a subset of the approximate point spectrum of T.

Proof. The last statement of the proposition is well-known ([C], Corollary XI.2.6) and the second statement follows from the first.

Let $\lambda \in \sigma_{ap}(T)$ and $\{x_\ell\}_{p=1}^\infty \subseteq \mathcal{H}$ be a sequence of unit vectors such that $(T - \lambda)x_\ell \to 0$. For $\ell \geq 0$ and $m \geq 0$, let $y_{\ell,m} = T^m x_\ell - \lambda^m x_\ell$. Now,

$$\lim_{\ell \to \infty} ||y_{\ell,m}|| = \lim_{\ell \to \infty} \left|\left| \left(\sum_{j=0}^{m-1} \lambda^j T^{m-1-j} \right) (T - \lambda) x_\ell \right|\right|$$

$$\begin{aligned} &\leq \left\| \sum_{j=0}^{m-1} \lambda^j T^{m-1-j} \right\| \lim_{\ell\to\infty} ||(T-\lambda)x_\ell|| \\ &= 0 . \end{aligned}$$

Therefore,

$$\langle T^{*^n} T^m x_\ell, x_\ell \rangle = \langle T^m x_\ell, T^n x_\ell, T^n x_\ell \rangle \to \overline{\lambda}^n \lambda^m .$$

By taking linear combinations, we see that

$$\langle p(T)x_\ell, x_\ell \rangle \longrightarrow p(\lambda, \overline{\lambda}) .$$

Thus $p(\lambda,\overline{\lambda}) \in W(p(T))^-$ which completes the proof of Proposition 1.12.

Before continuing, we make three observations. Firstly, note that if Proposition 1.12 is applied to the polynomial $p(x,y) = yx - 1$, then one can deduce that the approximate point spectrum of an isometry lies on the unit circle and that, by using elementary facts about the spectrum of an operator (e.g., in [C]), the spectrum is a subset of $\partial\mathbf{D}$ or equals $\mathbf{D}^-$ where $\mathbf{D}$ is the open unit disk in the complex plane. Summarizing, the spectral picture [P] of a finitely cyclic isometry can be deduced from Proposition 1.12 via the solution to the equation $\overline{\lambda}\lambda - 1 = 0$ for $\lambda \in \mathbf{C}$ without reference to function theory. Proposition 1.12 provides the key step for analyzing the spectral picture of a root of a polynomial in $\mathbf{C}[x,y]$. Secondly, an alternate proof of Proposition 1.12 consists of the observations that, with the setup of Proposition 1.12, (1.8) implies that if λ is an eigenvalue of T, then $p(\lambda,\overline{\lambda}) \in W(p(T))$ and the existence of a Hilbert space $\mathcal{K}$ and a unital *-representation $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ such that

$$\sigma_{ap}(R) = \sigma_p(\pi(R)) = \sigma_{ap}(\pi(R)) \quad \text{and} \quad W(R)^- = W(\pi(R)) . \tag{1.13}$$

The existence of such *-representations π is proven in [Ber]. Thirdly, note that for $p \in \mathbf{C}[x,y]$, $N \in \mathcal{L}(\mathcal{H})$ and N normal, $p(N) = 0$ if and only if $\sigma(N) \subseteq \{\lambda :$

$p(\lambda, \overline{\lambda}) = 0\}$.

Proposition 1.12 motivates the following definition.

Definition 1.14. *For $p \in \mathbf{C}[x,y]$, let $\Lambda_p = \{\lambda \in \mathbf{C} : p(\lambda, \overline{\lambda}) = 0\}$.*

Note that Λ_p is a real algebraic variety, $\Lambda_p = \Lambda_{p^\vee}$ and $\Lambda_p = \Lambda_{\frac{p+p^\vee}{2}} \cap \Lambda_{\frac{p-p^\vee}{2i}}$ for $p \in \mathbf{C}[x,y]$.

If T is an $n \times n$ matrix, $p \in \mathbf{C}[x]$ and p is the minimal polynomial of T, then one can deduce from p not only the spectrum of T, but also the order of some of the Jordan cells in the Jordan decomposition of T. Both of these properties are reflected in the meromorphic function $R : \sigma(T)^c \to \mathcal{L}(\mathcal{H})$ defined by $R(\lambda) = (\lambda - T)^{-1}$. The function R is analytic on $\sigma(T)^c$ and has a pole at each element of $\sigma(T)$ (i.e., at each eigenvalue). The order of each pole equals the order of some corresponding Jordan cell. Finally, if R has a pole of order n at λ_0, then $||R(\lambda)||$ varies as $|\lambda - \lambda_0|^{-n}$ for λ near λ_0. One can deduce an inequality on the norm of $(\lambda - T)^{-1}$ for an operator $T \in \mathcal{L}(\mathcal{H})$ which is a root of a polynomial.

Proposition 1.15. *Let $K \subseteq \mathbf{C}$, K be compact, $p \in \mathbf{C}[x,y]$ and $T \in \mathcal{L}(\mathcal{H})$. If $p(T) = 0$, then there exists $M > 0$ such that*

$$|p(\lambda, \overline{\lambda})| \|h\| \leq M \|(T - \lambda)h\|$$

for $h \in \mathcal{H}$ and $\lambda \in K$. In particular,

$$\|(T - \lambda)^{-1}\| \leq \frac{M}{| \, p(\lambda, \overline{\lambda}) \, |} \tag{1.16}$$

for $\lambda \in K$ and $\lambda \notin \sigma(T)$.

Proof. For $\lambda \in \mathbf{C}$, $m \geq 0$ and $n \geq 0$, let $c_{mn}(\lambda) \in \mathbf{C}$ such that

$$p(x + \lambda, y + \overline{\lambda}) = \sum_{m,n} c_{mn}(\lambda) y^n x^m \, .$$

Let

$$M_{mn} = \sup\{|c_{mn}(\lambda)| : \lambda \in K\}$$

and

$$P_m = \sup\{||(T-\lambda)^m|| : \lambda \in K\}.$$

Since $c_{mn}(\lambda)$ and $||(T-\lambda)^m||$ are continuous functions of λ and K is compact, $M_{mn} < \infty$ and $P_m < \infty$. Therefore, for $h \in \mathcal{H}$,

$$\begin{aligned}
|p(\lambda,\overline{\lambda})|||h||^2 &= |<(p(T)-p(\lambda,\overline{\lambda}))h, h>| \\
&= |<(p(x+\lambda, y+\overline{\lambda})(T-\lambda)-p(\lambda,\overline{\lambda}))h, h>| \\
&= \left|\sum_{m+n\geq 1} c_{mn}(\lambda) <(T-\lambda)^m h,\ (T-\lambda)^n h>\right| \\
&\leq \sum_{m+n\geq 1} |c_{mn}(\lambda)|\ ||(T-\lambda)^m h||\ ||(T-\lambda)^n h|| \\
&\leq \left(\sum_{m\geq 1} M_{m0}P_{m-1} + \sum_{m\geq 0, n\geq 1} M_{mn}P_mP_{n-1}\right) ||(T-\lambda)h||||h|| \\
&= M||(T-\lambda)h||||h||
\end{aligned}$$

where $M = \sum_{m\geq 1} M_{m0}P_{m-1} + \sum_{m\geq 0,n\geq 1} M_{mn}P_mP_{n-1}$. Since $c_{mn}(\lambda) \equiv 0$ for m, n sufficiently large, $M_{mn} = 0$ for m, n sufficiently large and $M < \infty$. This completes the proof of Proposition 1.15.

The following proposition shows that the right hand side of the resolvent inequality (1.16) can be replaced by $C/|\lambda - \lambda_0|^n$ for λ near λ_0 where C and n are positive.

Proposition 1.17. *Let $\lambda_0 \in \mathbf{C}$, $p(x,y) = \sum_{m+n\geq 1} c_{mn}(y-\overline{\lambda}_0)^n(x-\lambda_0)^m \in \mathbf{C}[x,y]$, $M = \min\{m+n : c_{mn} \neq 0\}$ and $f(z) = \sum_{m+n=m} c_{mn}z^{m-n}$ for $z \in \partial\mathbf{D}$. If $K \subseteq \partial\mathbf{D}, K$ is compact and $K \cap f^{-1}(\{0\}) = \phi$, then there exist $C > 0$ and $\epsilon > 0$ such that* *whenever* $\lambda \neq \lambda_0, |\lambda - \lambda_0| < \epsilon$ *and* $(\lambda - \lambda_0)/|\lambda - \lambda_0| \in K$

$$\frac{1}{|p(\lambda, \overline{\lambda})|} \leq \frac{C}{|\lambda - \lambda_0|^M}\,. \tag{1.18}$$

Proof. Let $\delta = \inf\{|f(z)| : z \in K\}$. Since K is compact and $K \cap f^{-1}(\{0\}) = \phi, \delta > 0$. Let $C = 2/\delta$ and $\epsilon > 0$ be such that

$$\sum_{m+n>M} |c_{mn}|\epsilon^{m+n-M} \leq \delta/2\,. \tag{1.19}$$

If $\lambda \neq \lambda_0$, $|\lambda - \lambda_0| < \epsilon$ and $(\lambda - \lambda_0)/|\lambda - \lambda_0| \in K$, then

$$\begin{aligned} |p(\lambda, \overline{\lambda})| &\geq \left|\sum_{m+n=M} c_{mn}(\lambda - \lambda_0)^m(\overline{\lambda} - \overline{\lambda}_0)^n\right| - \left|\sum_{m+n>M} c_{mn}(\lambda - \lambda_0)^m(\overline{\lambda} - \overline{\lambda}_0)^n\right| \\ &\geq |\lambda - \lambda_0|^M |f\left(\frac{\lambda - \lambda_0}{|\lambda - \lambda_0|}\right)| - \sum_{m+n>M} |c_{mn}||\lambda - \lambda_0|^{m+n} \\ &\geq |\lambda - \lambda_0|^M \left(\delta - \sum_{m+n>M} |c_{mn}||\lambda - \lambda_0|^{m+n-M}\right) \\ &\geq |\lambda - \lambda_0|^M/C\,. \end{aligned}$$

This establishes (1.18) and completes the proof of Proposition 1.17.

The following proposition is a variation of Theorem 6.3 in [R-R2] which, together with (1.16) and (1.18), proves that a root of a nonzero polynomial has hyperinvariant subspaces for each exposed arc of the spectrum. Since the proof of this variation differs from the proof of Theorem 6.3 only slightly, we will describe how to alter the proof of Theorem 6.3, rather than write out the complete proof. We begin with several definitions to be used in the hypothesis and proof of the proposition.

A *smooth Jordan arc* is a one-to-one function $z(t) = x(t) + iy(t)$ from (0, 1) into $\mathbf{C}$ such that d^2z/dt^2 exists everywhere in (0,1). If $K \subseteq \mathbf{C}$ and K is compact, then J is an *exposed arc* of K if there exists an open disk $\mathcal{D}$ such that $\mathcal{D} \cap K = J$ and J is a smooth Jordan arc.

Definition 1.20. *Let $T \in \mathcal{L}(\mathcal{H})$, C be an exposed arc of $\sigma(T)$, $\mathcal{D}$ be an open disk such that $\mathcal{D} \cap \sigma(T) = C$ and $\mathcal{D}_1$ and $\mathcal{D}_2$ be the connected open subsets of $\mathcal{D}$ such that $\mathcal{D} \backslash C = \mathcal{D}_1 \cup \mathcal{D}_2$. Let $\epsilon > 0$, k be a positive integer and $z_0 \in C$. $\underline{C \text{ is exposed to order } k \text{ from both sides at } z_0 \text{ for } T}$ if there exists $A > 0$ and two line segments L_1 and L_2 containing z_0 such that $L_1 \backslash \{z_0\} \subseteq \mathcal{D}_1, L_2 \backslash \{z_0\} \subseteq \mathcal{D}_2$ and*

$$||(z-T)^{-1}|| \leq \frac{A}{|z-z_0|^k} \tag{1.21}$$

for $z \in (L_1 \cup L_2) \backslash \{z_0\}$.

Let us agree to say that, for k a positive integer, C is exposed to order k from both sides for T if for every $z_0 \in C$, C is exposed to order k from both sides at z_0 for T.

Proposition 1.22. *Let $T \in \mathcal{L}(\mathcal{H})$, $k > 0$ and C be an exposed arc of $\sigma(T)$. Suppose that C is exposed to order k from both sides for T. T has a hyperinvariant subspace $\mathcal{N}$ such that $\sigma(T|\mathcal{N}) \subseteq C^-$.*

Proof. As previously mentioned, the proof of the existence of a hyperinvariant subspace for T differs from the proof of Theorem 6.3 in [R-R2] only slightly. We list these differences below.

1. In the first paragraph, assume that $|g'(x)| < \tan(\pi/5)$ rather than $|g'(x)| < \tan\left(\frac{\pi}{5k}\right)$.

2. In the second paragraph, construct the polygon $\Gamma_1(J)$ using the hypothesis that C is exposed from both sides at each point. Let $A > 0$ and L_1, L_2, L_3 and L_4 be closed line segments such that $z_1 \in L_1, z_1 \in L_2, z_2 \in L_3, z_2 \in L_4, L_1 \backslash \{z_1\} \subseteq \mathcal{D}_1, L_2 \backslash \{z_1\} \subseteq \mathcal{D}_2, L_3 \backslash \{z_2\} \subseteq \mathcal{D}_1, L_4 \backslash \{z_2\} \subseteq \mathcal{D}_2, L_1 \cap L_3 = \phi, L_2 \cap L_4 = \phi$,

$$||(z-T)^{-1}|| \leq A/|z-z_1|^k \quad z \in L_1 \cup L_2, z \neq z_1 \tag{1.23}$$

and

$$(1.24) \qquad ||(z-T)^{-1}|| \le A/|z-z_2|^k \quad z \in L_3 \cup L_4, z \neq z_2 .$$

It can be easily seen that one may connect the endpoints of L_1 and L_2 (resp., L_3 and L_4) which lie in $\mathcal{D}_1$ (resp., $\mathcal{D}_2$) with a polygonal path lying in $\mathcal{D}_1$ (resp., $\mathcal{D}_2$). Let $\Gamma_1(J)$ be the polygon constructed from L_1, L_2, L_3, L_4 and the two polygonal paths mentioned above.

3. The function $m(z)$ in the fifth paragraph should be changed to $m(z) = (z - \lambda_1)^k(z-\lambda_2)^k$.

4. The second displayed inequality in the sixth paragraph should be changed to

$$\begin{aligned} ||m(w)(R_n(w) - R_m(w))|| &\le |m(w)|||(w-T)^{-1}||\ ||h_n - h_m|| \\ &\le |(w-\lambda_1)^k(w-\lambda_2)^k|\frac{A}{|w-\lambda_1|^k}\ ||h_n - h_m|| \\ &= A|w-\lambda_2|^k||h_n - h_m|| \\ &\le C||h_n - h_m|| \end{aligned}$$

where A is an in (1.23) and $C = A\sup\{|w-\lambda_2|^k : w \in L_1 \cup L_2\}$.

5. The seventh paragraph should be omitted.

6. The function $m(z)$ in the eleventh paragraph should by changed to $m(z) = (z-z_1)^k(z-z_2)^k$.

These changes to the proof of Theorem 6.3 in [R-R2] complete the proof of the existence of a hyperinvariant subspace $\mathcal{N}$. The fact that $\sigma(T|\mathcal{N}) \subseteq C^-$ follows from Proposition 6.2 of [R-R2]. This completes the proof of Proposition 1.22.

The following corollary combines Proposition 1.22 and the Riesz Decomposition Theorem.

Corollary 1.25. *Let $T \in \mathcal{L}(\mathcal{H})$ and C be an exposed arc of $\sigma(T) \cup C$. If $\sigma(T) \cap C \neq \phi$ and for every subset X of $\sigma(T) \cap C$ which is a connected component and an arc, X is exposed to some order from both sides for T, then T has a hyperinvariant subspace $\mathcal{N}$ such that $\sigma(T|\mathcal{N}) \subseteq C^-$.*

Proof. If $\sigma(T) \cap C$ contains an open arc X, then X is an exposed arc for $\sigma(T)$ and Proposition 1.22 implies the existence of a subspace $\mathcal{N}$ hyperinvariant for T such that $\sigma(T|\mathcal{N}) \subseteq X^- \subseteq C^-$.

Suppose that $\sigma(T) \cap C$ does not contain any arcs. Let $z : (0,1) \to \mathbf{C}$ be a parameterization of C and $Y = z^{-1}(\sigma(T) \cap C)$. Let $y_2 \in Y$. Since $\sigma(T) \cap C$ does not contain any arcs, Y does not contain any arcs and there exist $y_1 \in (0, y_2)$ and $y_3 \in (y_2, 1)$ be such that $y_1 \notin Y$ and $y_3 \notin Y$.

Let $\sigma_1 = z([y_1, y_3]) \cap \sigma(T)$ and $\sigma_2 = \sigma(T) \backslash \sigma_1$. To see that σ_2 is closed, note that if $\mathcal{D}$ is an open disk such that $\mathcal{D} \cap (\sigma(T) \cup C) = C, x_n \in \sigma_2, x \in \sigma_1$ and $x_n \to x$, then $x \in \sigma(T)$ and $x_n \in \mathcal{D}$ for n sufficiently large. By passing to a subsequence if necessary, $z^{-1}(x_n)$ converge to an element t_0 and so $z(t_0) = x$. Since $z^{-1}(x_n) \in (0, z_1] \cup [z_3, 1)$, $x \in \mathcal{D}$ and z is one-to-one, $t_0 \in (0, z_1] \cup [z_3, 1)$ and

$$x = z(t_0) \in z((0, z_1] \cup [z_3, 1)) \cap z([z_1, z_3]) \cap \sigma(T) = z(\{z_1, z_3\}) \cap \sigma(T) = \phi.$$

This is a contradiction and so σ_2 is closed. The Riesz Decomposition theorem ([R-R2], Theorem 2.10 and Corollary 2.11) applies and yields a subspace $\mathcal{N}$ hyperinvariant for T such that $\sigma(T|\mathcal{N}) \subseteq C^-$.

The following corollary follows from Propositions 1.15 and 1.17 and Corollary 1.25.

Corollary 1.26. *Let $T \in \mathcal{L}(\mathcal{H})$ and $p \in \mathbf{C}[x,y]$. If p is not identically zero, $p(T) = 0$, C is an exposed arc of $\sigma(T) \cup C$ and $\sigma(T) \cap C$ is not empty, then T has* *a hyperinvariant subspace* $\mathcal{N}$ *such that* $\sigma(T|\mathcal{N}) \subseteq C^-$.

Chapter 2

Rosenblum's Theorem, von Neumann algebras, and roots of hereditary polynomials

In this chapter, we continue our investigation of roots of hereditary polynomials via applications of Rosenblum's Theorem and introduce the facts required from the theory of von Neumann algebras which will be used in the next chapter.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $B \in \mathcal{L}(\mathcal{H})$, $A \in \mathcal{L}(\mathcal{K})$ and $p \in \mathbf{C}[x, y]$. Let $\Phi_{A,B,p}$ be the linear transformation on $\mathcal{L}(\mathcal{H}, \mathcal{K})$ defined by

$$(2.1) \qquad \Phi_{A,B,p}(X) = \sum_{m,n} p^\wedge(m,n) A^n X B^m, \qquad X \in \mathcal{L}(\mathcal{H}, \mathcal{K}) .$$

$\Phi_{A,B,p}$ is clearly continuous (in X) and a theorem of Rosenblum's ([R-R2], Lemma 0.11 and Theorem 0.12) shows that

$$(2.2) \qquad \sigma(\Phi_{A,B,p}) \subseteq \{p(b,a) : b \in \sigma(B), a \in \sigma(A)\}.$$

Note that if $1_{\mathcal{H}}$ denotes the identity map on $\mathcal{L}(\mathcal{H})$, then $\Phi_{T^*,T,p}(1_{\mathcal{H}}) = p(T)$ for $T \in \mathcal{L}(\mathcal{H})$ and $p \in \mathbf{C}[x, y]$ and that

$$(2.3) \qquad \Phi_{A,B,p+q}(X) = \Phi_{A,B,p}(X) + \Phi_{A,B,q}(X),$$

$$(2.4) \qquad \Phi_{A,B,cp}(X) = c\Phi_{A,B,p}(X)$$

and

$$\Phi_{A,B,p^\vee}(X) = (\Phi_{B^*,A^*,p}(X^*))^* \tag{2.5}$$

wherever $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $p \in \mathbf{C}[x,y]$, $q \in \mathbf{C}[x,y]$, $c \in \mathbf{C}$, $B \in \mathcal{L}(\mathcal{H})$, $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and $A \in \mathcal{L}(\mathcal{K})$.

The following proposition uses (2.2) to give a condition on $q \in \mathbf{C}[x,y]$ such that $(pq)(T) = 0$ implies $p(T) = 0$, where $p \in \mathbf{C}[x,y]$ and $T \in \mathcal{L}(\mathcal{H})$.

Proposition 2.6. *Let $p \in \mathbf{C}[x,y]$, $q \in \mathbf{C}[x,y]$ and $T \in \mathcal{L}(\mathcal{H})$. If $(pq)(T) = 0$ and $q(\lambda,\overline{\mu}) \neq 0$ for every $\lambda, \mu \in \sigma(T)$, then $p(T) = 0$.*

Proof. Since $p(x,y)q(x,y) = \sum_{m,n} q^\wedge(m,n) y^n p(x,y) x^m$, Proposition 1.2 implies that $(pq)(T) = \Phi_{T^*,T,q}(p(T))$. Since $(pq)(T) = 0, 0 \in \sigma(\Phi_{T^*,T,q})$ or $p(T) = 0$. By (2.2), that fact that $\sigma(T^*) = \{\overline{\mu} : \mu \in \sigma(T)\}$ and the hypothesis of the proposition, $0 \notin \sigma(\Phi_{T^*,T,q})$. Therefore, $p(T) = 0$ and the proof of Proposition 2.6 is complete.

In contrast to Proposition 2.6, the following proposition shows that if T is a root of pq and $p(T)$ is both positive and invertible, then $\sigma_{ap}(T) \subseteq \Lambda_q$ (see Definition 1.14) rather than just $\sigma_{ap}(T) \subseteq \Lambda_{pq}$ as guaranteed by Proposition 1.12.

Proposition 2.7. *Let $p \in \mathbf{C}[x,y], q \in \mathbf{C}[x,y]$ and $T \in \mathcal{L}(\mathcal{H})$. If $(pq)(T) = 0, p(T)$ is positive and $p(T)$ is invertible, then $\sigma_{ap}(T) \subseteq \Lambda_q$ and T is similar to a root of q.*

Proof. Let $S = p(T)^{\frac{1}{2}}$ and $R = STS^{-1}$. Since

$$T^{*n} p(T) T^m = S^* R^{*n} R^m S$$

for $m \geq 0$ and $n \geq 0$,

$$\begin{aligned} S^* q(R) S &= \sum_{m,n} q^\wedge(m,n) S^* R^{*n} R^m S \\ &= S^*((pq)(T))S \\ &= 0\,. \end{aligned}$$

Since S is invertible, $q(R) = 0$. By Proposition 1.12, $\sigma_{ap}(R) \subseteq \Lambda_q$. Since T is similar to R, $\sigma_{ap}(T) = \sigma_{ap}(R) \subseteq \Lambda_q$. This completes the proof of Proposition 2.7.

The following proposition and (2.2) prove that one can combine properties of a polynomial in $\mathbf{C}[x, y]$ and facts involving the spectrum of a root to show that certain invariant subspaces are actually reducing.

Proposition 2.8. *Let $p \in \mathbf{C}[x, y]$, $T \in \mathcal{L}(\mathcal{H})$, $\{\mathcal{M}, \mathcal{N}\}$ be a pair of complimentary invariant subspaces for T and $P_1, P_2 \in \mathcal{L}(\mathcal{H})$ be the idempotents defined by*

$$\begin{aligned} P_1(m+n) &= m \\ P_2(m+n) &= n \end{aligned} \qquad m \in \mathcal{M}, n \in \mathcal{N}. \tag{2.9}$$

If T has the block operator form

$$T = \begin{bmatrix} B & E \\ 0 & C \end{bmatrix} \tag{2.10}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $p(T) = 0$, then

$$E = BY - YC, \tag{2.11}$$

$$\Phi_{C^*,B,p}(Y^*) = 0 \tag{2.12}$$

and

$$\Phi_{B^*,C,p}(Y) = 0 \tag{2.13}$$

where $Y = P_{\mathcal{M}} P_1 | \mathcal{M}^\perp$ and $P_{\mathcal{M}} \in \mathcal{L}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.

Proof. By (2.9), P_1 and P_2 have the block operator forms

$$P_1 = \begin{bmatrix} 1 & Y \\ 0 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & -Y \\ 0 & 1 \end{bmatrix} \tag{2.14}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

Computation using (2.10) and (2.14) yields

$$\begin{aligned} \mathcal{N} &= \{\begin{bmatrix} -Yk \\ k \end{bmatrix} : k \in \mathcal{M}^{\perp}\} \qquad \text{and} \\ T\mathcal{N} &= \{\begin{bmatrix} (E - BY)\ell \\ C\ell \end{bmatrix} : \ell \in \mathcal{M}^{\perp}\}. \end{aligned}$$

Since $\mathcal{N}$ is invariant for T, $E - BY = -YC$. Therefore, (2.11) holds.

Now, for $m \geq 0$, computation using (2.10) and (2.11) yields

$$T^m = \begin{bmatrix} B^m & B^mY - YC^m \\ 0 & C^m \end{bmatrix}. \tag{2.15}$$

Therefore, computation using (2.15), (1.1) and (2.1) yields

$$p(T) = \begin{bmatrix} p(B) & p(B)Y - \Phi_{B^*,C,p}(Y) \\ Y^*p(B) - \Phi_{C^*,B,p}(Y^*) & * \end{bmatrix}. \tag{2.16}$$

Since $p(T) = 0$, $p(B) = 0$ and (2.12) and (2.13) hold. This completes the proof of Proposition 2.8.

Corollary 2.17. *Let $a \in \mathbf{C}$, $b \in \mathbf{C}$, $p \in \mathbf{C}[x,y]$, $T \in \mathcal{L}(\mathcal{H})$ and $\{\mathcal{M},\mathcal{N}\}$ be a pair of complimentary invariant subspaces for T. If $p(T) = 0$ and the set*

$$\{(\lambda, \mu) \in \sigma(T|\mathcal{M}) \times \sigma(T|\mathcal{N}) : ap(\lambda, \overline{\mu}) + b\overline{p(\mu, \overline{\lambda})} = 0\} \tag{2.18}$$

is empty, then $\mathcal{M} = \mathcal{N}^{\perp}$ and $\mathcal{M}$ reduces T.

Proof. Let T have the block operator form (2.10) with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ and Y be as in Proposition 2.8. Since $p(T) = 0$, (2.11), (2.12) and (2.13) hold. By (2.3), (2.4), (2.5), (2.12) and (2.13),

$$\Phi_{C^*,B,ap+bp^{\vee}}(Y^*) = 0.$$

Since $\sigma(C) = \sigma(T|\mathcal{N})$ ([R-R], Lemma 1), (2.2) and the emptiness of the set of (2.18) implies $Y = 0$. By (2.11), $E = 0$. Thus $\mathcal{M}$ reduces T and $\mathcal{M} = \mathcal{N}^{\perp}$, which completes the proof of Corollary 2.17.

Note that if both $a = 1$ and $b = 0$ or both $a = 0$ and $b = 1$, then the set (2.18) has the simpler forms

$$\{(\lambda, \mu) \in \sigma(T|\mathcal{M}) \times \sigma(T|\mathcal{N}) : p(\lambda, \overline{\mu}) = 0\}$$

or

$$\{(\lambda, \mu) \in \sigma(T|\mathcal{M}) \times \sigma(T|\mathcal{N}) : p(\mu, \overline{\lambda}) = 0\} .$$

If $p = p^{\vee}$, then these two sets are equal. Also note that

$$\{(\lambda, \mu) \in \sigma(T|\mathcal{M}) \times \sigma(T|\mathcal{N}) : p(\lambda, \overline{\mu}) = p(\mu, \overline{\lambda}) = 0\}$$

is a subset of the set (2.18).

If, in Proposition 2.6, Proposition 2.7 and Corollary 2.17, we replace "spectrum" with "essential spectrum" and roughly "equal to zero" with "compact", then three new statements hold. If T is an operator, then $\sigma_e(T)$ denotes the essential spectrum of T. These three statements may be proven using elementary facts concerning Calkin algebras and representations of, and spectral permanence for, C^*-algebras [C]. We omit these proofs.

Proposition 2.19. *Let $p \in \mathbf{C}[x,y]$, $q \in \mathbf{C}[x,y]$ and $T \in \mathcal{L}(\mathcal{H})$. If $(pq)(T)$ is compact and $q(\lambda, \overline{\mu}) \neq 0$ for every $\lambda, \mu \in \sigma_e(T)$,then $p(T)$ is compact.*

An operator $T \in \mathcal{L}(\mathcal{H})$ is *essentially positive* if $\pi(T)$ is positive where π is the projection of $\mathcal{L}(\mathcal{H})$ into the Calkin algebra on $\mathcal{H}$.

Proposition 2.20. *Let $p \in \mathbf{C}[x,y]$, $q \in \mathbf{C}[x,y]$ and $T \in \mathcal{L}(\mathcal{H})$. If $(pq)(T)$ is compact, $p(T)$ is essential positive and 0 is not in the essential spectrum of $p(T)$, then the essential spectrum of T is a subset of Λ_q.*

Proposition 2.21. *Let $a \in \mathbf{C}$, $b \in \mathbf{C}$, $p \in \mathbf{C}[x,y]$, $\mathcal{H}$ be a separable Hilbert space, $T \in \mathcal{L}(\mathcal{H})$ and $\{\mathcal{M},\mathcal{N}\}$ be a pair of complimentary invariant subspaces for T. If $p(T)$ is compact and the set*

$$\{(\lambda,\mu) \in \sigma_e(T|\mathcal{M}) \times \sigma_e(T|\mathcal{N}) : ap(\lambda,\overline{\mu}) + b\overline{p(\mu,\overline{\lambda})} = 0\} \tag{2.22}$$

is empty, then $P_{\mathcal{M}}T|\mathcal{M}^{\perp}$ is compact.

Proposition 2.8, Corollary 2.17 and Propositions 2.20 and 2.22 are rather restrictive since they presuppose the existence of a pair of complimentary invariant subspaces. For example, if A is self-adjoint and $\sigma(A)$ is not connected, then Corollary 2.17 and the Riesz Decomposition Theorem [R-R2] imply (without reference to the spectral theorem) that A has a reducing subspace. On the other hand, if A is self-adjoint and $\sigma(A)$ is an interval, then Corollary 2.17 does not apply. Indeed, if one had a nontrivial pair of complimentary invariant subspaces $\{\mathcal{M},\mathcal{N}\}$, then $\sigma(T|\mathcal{M}) \cap \sigma(T|\mathcal{N})$ would not be empty and if λ is in this intersection, then $p(\lambda,\overline{\lambda}) = 0$.

The following proposition is an implication of Rosenblum's Theorem and (together with the results of the next chapter) overcomes the above mentioned difficulties.

Proposition 2.23. *Let $p \in \mathbf{C}[x,y]$, $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $A \in \mathcal{L}(\mathcal{K})$, $B \in \mathcal{L}(\mathcal{H})$ and suppose that*

$$\begin{gathered}\{\mathcal{N}_k\}_{k=1}^{\infty} \text{ is a sequence of invariant subspaces for } B \text{ such that} \\ (\textstyle\bigcup_{k=1}^{\infty} \mathcal{N}_k)^{-} = \mathcal{H}\end{gathered} \tag{2.24}$$

and

$$\text{for each } k \geq 1,\ p(b,a) \neq 0 \text{ whenever } a \in \sigma(A) \text{ and } b \in \sigma(B|\mathcal{N}_k)\ . \tag{2.25}$$

If $\sum_{m,n} p^\wedge(m,n)A^nXB^m = 0$, *then* $X = 0$.

Proof. If $k \geq 1$, then B and X have the following block operator forms

$$B = \begin{bmatrix} B|\mathcal{N}_k & P_{\mathcal{N}_k}B|\mathcal{N}_k^\perp \\ 0 & P_{\mathcal{N}_k^\perp}B|\mathcal{N}_k^\perp \end{bmatrix} \in \mathcal{L}(\mathcal{N}_k \oplus \mathcal{N}_k^\perp) \tag{2.26}$$

and

$$X = \begin{bmatrix} X|\mathcal{N}_k & X|\mathcal{N}_k{}^\perp \end{bmatrix} \in \mathcal{L}(\mathcal{N}_k \oplus \mathcal{N}_k^\perp, \mathcal{K}). \tag{2.27}$$

Since $\sum p^\wedge(m,n)A^nXB^m = 0$, we find that by using (2.26) and (2.27),

$$\sum_{m,n} p^\wedge(m,n)A^n(X \mid \mathcal{N}_k)B_1^m = 0.$$

By (2.25) and (2.2), $X \mid \mathcal{N}_k = 0$. Therefore, $X(\mathcal{N}_k) = \{0\}$ for all $k \geq 1$. Thus, by (2.24), $X = 0$. This completes the proof of Proposition 2.23.

We have the following corollary which will be used in the next chapter with a few facts from the theory of von Neumann algebras. The required result involving von Neumann algebras will be given after the statement and proof of the corollary. If $\mathcal{H}$ is a Hilbert space and $X \subseteq \mathcal{L}(\mathcal{H})$, then X' denotes the commutator of X,

$$X' = \{R \in \mathcal{L}(\mathcal{H}) : RT = TR \quad \text{for each } T \in X\}\ .$$

Let $X'' = (X')'$ and $X''' = (X'')'$. It is well known that $X \subseteq X''$ and $X' = X'''$ for any set $X \subseteq \mathcal{L}(\mathcal{H})$ ([D],section I.1.1).

Corollary 2.28. *Let* $A \in \mathcal{L}(\mathcal{K})$, $B \in \mathcal{L}(\mathcal{H})$ *and* $\{\mathcal{N}_k\}$ *be as in (2.24). If, for each* $k \geq 1$,

$$\sigma(A) \cap \sigma(B|\mathcal{N}_k) = \phi\,, \tag{2.29}$$

then

$$\{A \oplus B, A^* \oplus B^*\}' = \{X \oplus Y : X \in \{A, A^*\}' \text{ and } Y \in \{B, B^*\}'\} \tag{2.30}$$

and

$$(2.31) \qquad 1 \oplus 0, 0 \oplus 1 \in \{A \oplus B, A^* \oplus B^*\}' \cap \{A \oplus B, A^* \oplus B^*\}'' .$$

Proof. For $X \in \mathcal{L}(\mathcal{K})$, $Y \in \mathcal{L}(\mathcal{H})$, $Z \in \mathcal{L}(\mathcal{K},\mathcal{H})$ and $W \in \mathcal{L}(\mathcal{H},\mathcal{K})$,

$$(2.32) \qquad \begin{bmatrix} X & Z \\ W & Y \end{bmatrix} \in \{A \oplus B, A^* \oplus B^*\}'$$

if and only if

$$(2.33) \qquad \begin{gathered} AX = XA, BY = YB, BW = WA, ZB = AZ, \\ A^*X = XA^*, B^*Y = YB^*, B^*W = WA^* \text{ and } ZB^* = A^*Z . \end{gathered}$$

Using (2.29) and an application of Proposition 2.23 to $p(x,y) = x - y$, (2.33) is equivalent to

$$(2.34) \qquad X \in \{A, A^*\}' \ , \ Y \in \{B, B^*\}', \ W = 0 \ \text{ and } \ Z = 0.$$

Therefore, (2.30) follows.

Since the zero and identity transformations commute with every operator, $1 \oplus 0, 0 \oplus 1 \in \{A \oplus B, A^* \oplus B^*\}'$ and $1 \oplus 0$ and $0 \oplus 1$ commute with $X \oplus Y$ whenever $X \in \{A, A^*\}'$ and $Y \in \{B, B^*\}'$. Thus, by (2.30), $1 \oplus 0, 0 \oplus 1 \in \{A \oplus B, A^* \oplus B^*\}''$. Therefore, (2.31) holds. This completes the proof of Corollary 2.28.

We recall the following definition and theorem from [D].

Definition 2.35 [D]. *A <u>von Neumann algebra in $\mathcal{H}$</u> is a *-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A} = \mathcal{A}''$. The* **center** *$Z$ of a von Neumann algebra $\mathcal{A}$ is $\mathcal{A} \cap \mathcal{A}'$. A <u>factor</u> is a von Neumann algebra whose center is the collection of scalar multiples of the identity.*

Before continuing, note that (2.30) of Corollary 2.27 implies that $\{A \oplus B, A^* \oplus B^*\}''$ is **not** a factor (under the stated hypothesis).

In sections II.1, II.2, II.3.1, II.3.2, and II.3.3 of [D] , the theory of direct integrals of Hilbert spaces, operators and von Neumann algebras is developed.

We shall need a result from section II.6.2 of [D].

Theorem 2.36 [D]. *Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{A}$ be a von Neumann algebra in $\mathcal{H}$. There exists a compact metrizable space Z, a positive measure ν on Z of support Z, a ν-measurable field $\zeta \to \mathcal{H}(\zeta)$ of non-zero complex Hilbert spaces over Z, a ν-measurable field $\zeta \to \mathcal{A}(\zeta)$ of factors in the $\mathcal{H}(\zeta)'s$, and a Hilbert space isomorphism of $\mathcal{H}$ onto $\int_\oplus \mathcal{H}(\zeta)d\nu(\zeta)$ which transforms $\mathcal{A}$ into $\int_\oplus \mathcal{A}(\zeta)d\nu(\zeta)$.*

Chapter 3

Families, Maximal Invariant Subspaces and Several Examples of Roots

It is easy to see that if $T \in \mathcal{L}(\mathcal{H})$, K is a compact subset of $\mathbf{R}$ and $\mathcal{M}_0$ is an invariant subspace of T such that $T|\mathcal{M}_0$ is self-adjoint and $\sigma(T|\mathcal{M}_0) \subseteq K$, then there exists a subspace $\mathcal{M} \subseteq \mathcal{H}$ which is maximal with respect to the conditions $\mathcal{M}_0 \subseteq \mathcal{M}$, $\mathcal{M}$ is invariant for T, $T|\mathcal{M}$ is self-adjoint and $\sigma(T|\mathcal{M}) \subseteq K$. In this chapter, we generalize the above observation from the context of self-adjoint operators whose spectrum lies in a prescribed set to the more general context of families of operators (which are defined below) and show how this result (and the results of chapters one and two) allow us to classify

$$\{R : p(R) = 0\}$$

for many polynomials p. We finish the chapter by classifying subnormal roots.

We now state the definition of *family of operators* from [Ag1] or [Ag8].

Definition 3.1. *Let $\mathcal{F}$ be a collection of operators. $\mathcal{F}$ is a* <u>*family of operators*</u> *if*

$$\bigoplus_{\alpha \in I} T_\alpha \in \mathcal{F} \text{ whenever } I \text{ is an index set and } T_\alpha \in \mathcal{F} \text{ for every } \alpha \in I \tag{3.2}$$

$$\text{(3.3)} \quad \begin{array}{c}\pi(T) \in \mathcal{F} \text{ whenever } \mathcal{H} \text{ is a Hilbert space}, T \in \mathcal{F} \cap \mathcal{L}(\mathcal{H}), \mathcal{K} \text{ is a Hilbert} \\ \text{space and } \pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}) \text{ is a unital } * -\text{representation}\end{array}$$

and

$$\text{(3.4)} \qquad T|\mathcal{M} \in \mathcal{F} \text{ whenever } T \in \mathcal{F} \text{ and } \mathcal{M} \text{ is an invariant suspace for } T.$$

If $\mathcal{F}$ is a collection of operators satisfying (3.2) ((3.3) or (3.4)), then we say that $\mathcal{F}$ is *closed with respect to direct sums* (*unital *-representations* or *restrictions to invariant subspaces*, resp.).

Before continuing, we make two observations about families of operators. Let $\mathcal{F}$ be a family of operators. $\mathcal{F}$ is bounded (i.e., $\sup\{||R|| : R \in \mathcal{F}\} < \infty$) since $\mathcal{F}$ is closed with respect to direct sums. $\mathcal{F}$ is closed under conjugation by Hilbert space isomorphisms (i.e., $U^*TU \in \mathcal{F}$ whenever $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $T \in \mathcal{L}(\mathcal{K}) \cap \mathcal{F}$ and $U \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a Hilbert space isomorphism) since $\mathcal{F}$ is closed under unital *-representations and $\pi(X) = U^*XU$ for $X \in \mathcal{L}(\mathcal{K})$ is a unital *-representation if $U \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a Hilbert space isomorphism.

We now give a short list of examples to illustrate the concept of a family of operators. For a longer list of examples and more facts concerning families, see [Ag8].

Example 3.5 For $\sigma \geq 0$, it is easy to see that $\{R : R \text{ is subnormal and } ||R|| \leq \sigma\}$ is a family of operators.

Example 3.6 For $\sigma \geq 0$ and K a polynomially convex subset of $\mathbf{C}$, the fact that $\{R : R \text{ is subnormal}, ||R|| \leq \sigma, \text{ and } \sigma(R) \subseteq K\}$ is a family of operators follows from the facts that both $\sigma(R|\mathcal{M})$ is a subset of the polynomially convex hull of $\sigma(R)$ whenever R is an operator and $\mathcal{M}$ is an invariant subspace for R and the well-known equality

$$||(R - \lambda)^{-1}|| = \frac{1}{\operatorname{dist}(\lambda, \sigma(R))}$$

where $\lambda \notin \sigma(R)$ and $\text{dist}(\lambda, \sigma(R))$ denotes the distance between λ and $\sigma(R)$.

Example 3.7 If we replace the word "subnormal" with "hyponormal" in Examples 3.5 and 3.6, then the resulting collections are families.

Example 3.8 For $\sigma \geq 0$ and $p \in \mathbf{C}[x, y]$, it is easy to see that

$$\{R : p(R) = 0 \quad \text{and} \quad ||R|| \leq \sigma\} \tag{3.9}$$

is a family of operators.

Example 3.10 The collection

$$\mathcal{F} = \{R : ||R|| \leq 1 \quad \text{and } \sigma(R) = \{0\}\}$$

is *not* a family of operators. Indeed, if S is the unilateral shift on Hardy space, H^2, and $\mathcal{M}_n$ is the span of $\{1, \ldots, z^n\}$, then $\mathcal{M}_n$ is invariant for S^*, $S^*|\mathcal{M}_n \in \mathcal{F}$ but $\bigoplus_{n=1}^{\infty} (S^*|\mathcal{M}_n) \notin \mathcal{F}$, since its spectrum is not $\{0\}$.

The following proposition describes a key difference between Examples 3.6 and 3.10. For a family of operators $\mathcal{F}$ and K a polynomially convex subset of $\mathbf{C}$, let

$$\mathcal{F}_K = \{R : R \in \mathcal{F} \quad \text{and} \quad \sigma(R) \subseteq K\}. \tag{3.11}$$

As illustrated by Example 3.10, $\mathcal{F}_K$ need not be a family of operators.

Proposition 3.12. *Let $\mathcal{F}$ be a family of operators and K be a polynomially convex subset of $\mathbf{C}$. $\mathcal{F}_K$ is a family of operators if and only if*

$$\sup\{||(R - \lambda)^{-1}|| : R \in \mathcal{F}_K\} < \infty \tag{3.13}$$

for every $\lambda \notin K$.

Proof. Suppose $\mathcal{F}_K$ is a family and (3.13) does not hold. There exist $\lambda \notin K$ and $R_n \in \mathcal{F}$ such that $||(R_n - \lambda)^{-1}|| \geq n$ for each $n \geq 1$. Let $R = \bigoplus_{n=1}^{\infty} R_n$. Since $\mathcal{F}_K$ is a family of operators, $R \in \mathcal{F}_K$ and so $\lambda \notin \sigma(R)$. But

$$||(R-\lambda)^{-1}|| \geq ||(R_n-\lambda)^{-1}|| \geq n$$

for each $n \geq 1$ which contradicts $\lambda \notin \sigma(R)$. Therefore, if $\mathcal{F}_K$ is a family of operators, then (3.13) holds.

Suppose now that (3.13) holds. Since $\mathcal{F}$ is a family and $\sigma(\pi(R)) \subseteq \sigma(R)$ for any operator $R \in \mathcal{L}(\mathcal{H})$ and any unital *-representation π, $\mathcal{F}_K$ is closed with respect to unital *-representations. Since $\mathcal{F}$ is a family of operators and K is polynomially convex, $\mathcal{F}$ is closed with respect to restrictions to invariant subspaces. To show that $\mathcal{F}_K$ is closed with respect to direct sums, let I be an index set, $T_\alpha \in \mathcal{F}_K \cap \mathcal{L}(\mathcal{H}_\alpha)$ for each $\alpha \in I$ and $T = \bigoplus_{\alpha \in I} T_\alpha \in \mathcal{L}(\bigoplus_{\alpha \in I} \mathcal{H}_\alpha)$. Since $\mathcal{F}$ is a family, $T \in \mathcal{F}$. Now, let $\lambda \notin K$ and

$$M = \sup\{||(R-\lambda)^{-1}|| : R \in \mathcal{F}_K\} \ .$$

Since $\lambda \notin \sigma(T_\alpha)$ for each $\alpha \in I$, $T-\lambda$ is injective. Since (3.13) holds, $M < \infty$ and $\bigoplus_{\alpha \in I} (T_\alpha - \lambda)^{-1}$ is bounded. Since

$$(T-\lambda)\left(\bigoplus_{\alpha \in I}(T_\alpha-\lambda)^{-1}\right) = 1 \ ,$$

$T-\lambda$ is onto. Since $T-\lambda$ is one-to-one, onto and linear, $T-\lambda$ is invertible by the open mapping theorem. Therefore, $\sigma(T) \subseteq K$ and $T \in \mathcal{F}_K$. Thus, if (3.13) holds, then $\mathcal{F}_K$ is a family. This completes the proof of Proposition 3.12.

Before proceeding to the first main result of this chapter (Theorem 3.16) we need the following unpublished result of Jim Agler. We say that a collection of operators is *closed in the strong operator topology* (SOT) if $T \in \mathcal{F}$ whenever T_α is a net converging to T in the strong operator topology and $T_\alpha \in \mathcal{F}$ for each α.

Proposition 3.14. *If $\mathcal{F}$ is a family of operators, then $\mathcal{F}$ is closed in the strong operator topology.*

Proof. Let $\mathcal{F}$ be a family of operators, $\mathcal{H}$ be a Hilbert space and $\{R_\alpha\}$ be a net in $\mathcal{F} \cap \mathcal{L}(\mathcal{H})$ converging to $R \in \mathcal{L}(\mathcal{H})$ in the strong operator topology. We wish to show that $R \in \mathcal{F}$.

By Theorem 2.6 of [Ag2], there exists a collection $X \subseteq \bigcup_{n=1}^{\infty} \mathbf{C}^{n,n} \otimes \mathbf{C}[x,y]$ such that $T \in \mathcal{F}$ if and only if $p(T) \geq 0$ for every $p \in S$. Therefore, we need only show $p(R) \geq 0$ for every $p \in S$. This will follow by a demonstration that $p(R_\alpha)$ converges to $p(R)$ in the weak operator topology (WOT) for every $p \in \bigcup_{n=1}^{\infty} \mathbf{C}^{n,n} \otimes \mathbf{C}[x,y]$.

For $p \in \mathbf{C}[x,y]$ and $h, k \in \mathcal{H}$, note that

$$\begin{aligned} \langle p(R_\alpha)h, k\rangle &= \sum_{m,n} p^\wedge(m,n)\langle R_\alpha^{*n} R_\alpha^m h, k\rangle \\ &= \sum_{m,n} p^\wedge(m,n)\langle R_\alpha^m h, R_\alpha^n k\rangle \\ &\to \sum_{m,n} p^\wedge(m,n)\langle R^m h, R^n k\rangle \\ &= \langle p(R)h, k\rangle . \end{aligned}$$

Therefore,

$$p(R_\alpha) \longrightarrow p(R) \text{ in } WOT, \quad p \in \mathbf{C}[x,y] . \tag{3.15}$$

Fix $N \geq 1$ and B a basis of $\mathbf{C}^N$. Let $p \in \mathbf{C}^{N,N} \otimes \mathbf{C}[x,y]$. There exist $K \geq 1$, $M_k \in \mathbf{C}^{N,N}$ for $1 \leq k \leq K$ and $p_k \in \mathbf{C}[x,y]$ for $1 \leq k \leq K$ such that $p = \sum_{k=1}^{K} M_k \otimes p_k$. For $h = \sum_{b \in B} b \otimes h_b$ and $m = \sum_{b \in B} b \otimes m_b$ in $\mathbf{C}^N \otimes \mathcal{H}$,

$$\begin{aligned} \langle p(R_\alpha)h, m\rangle &= \sum_{b_1, b_2 \in B, 1 \leq k \leq K} \langle M_k \otimes p_k(R_\alpha) b_1 \otimes h_{b_1}, b_2 \otimes m_{b_2}\rangle \\ &= \sum_{k, b_1, b_2} \langle M_k b_1, b_2\rangle \langle p_k(R_\alpha) h_{b_1}, m_{b_2}\rangle \end{aligned}$$

$$\begin{aligned} &\to \sum_{k,b_1,b_2} \langle M_k b_1, b_2\rangle \langle p_k(R) h_{b_1}, m_{b_2}\rangle \qquad (3.15)\\ &= \langle p(R)h, m\rangle . \end{aligned}$$

Thus $p(R_\alpha)$ converges to $p(R)$ in the weak operator topology. This completes the proof of Proposition 3.14.

Theorem 3.16. *Let $\mathcal{F}$ be a family of operators and $T \in \mathcal{L}(\mathcal{H})$.*

(a) If $\mathcal{M}_0$ is an invariant subspace of T such that $T|\mathcal{M}_0 \in \mathcal{F}$, then there exists $\mathcal{M} \subseteq \mathcal{H}$ such that $\mathcal{M}_0 \subseteq \mathcal{M}$ and is maximal with respect to the conditions that $\mathcal{M}$ is a subspace of $\mathcal{H}$, $\mathcal{M}$ is invariant for T and $T|\mathcal{M} \in \mathcal{F}$.

(b) If I is a totally ordered set, $\{M_\alpha\}_{\alpha\in I}$ is a collection of invariant subspaces of T such that $T|\mathcal{M}_\alpha \in \mathcal{F}$ for each $\alpha \in I$ and such that $\mathcal{M}_\alpha \subseteq \mathcal{M}_\beta$ whenever $\alpha, \beta \in I$ and $\alpha \leq \beta$, then $\mathcal{M} = \left(\bigcup_{\alpha\in I} \mathcal{M}_\alpha\right)^-$ is an invariant subspace for T and $T|\mathcal{M} \in \mathcal{F}$.

Proof. Note that (a) follows from (b) by an application of Zorn's Lemma. To prove (b), let I and $\mathcal{M}_\alpha$ be as in the statement of the theorem. We first show the technical fact that there exists $\lambda \in \mathbf{C}$ such that $\lambda 1_{\mathcal{J}} \in \mathcal{F}$ for every Hilbert space $\mathcal{J}$ (where $1_{\mathcal{J}}$ is the identity on $\mathcal{J}$). Let $\alpha_0 \in I$ and $\lambda \in \sigma_{ap}(T|\mathcal{M}_{\alpha_0})$. Let $\mathcal{K}$ be a Hilbert space and $\pi : \mathcal{L}(\mathcal{M}_{\alpha_0}) \to \mathcal{L}(\mathcal{K})$ be a unital *-representation satisfying (1.13). Let $\mathcal{N}$ be a one dimensional subspace of $\ker(\pi(T|\mathcal{M}_{\alpha_0}) - \lambda)$. Since every Hilbert space is unitarily equivalent to the direct sum of copies of $\mathcal{N}$ and $\mathcal{F}$ is a family of operators, $\lambda 1_{\mathcal{J}} \in \mathcal{F}$ for every Hilbert space $\mathcal{J}$.

Let the set

$$\mathcal{S} = \{\mathcal{M} : \mathcal{M} \text{ is a subspace of } \mathcal{H}, \mathcal{M} \text{ is invariant for } T \text{ and } T|\mathcal{M} \in \mathcal{F}\}$$

be ordered by inclusion. Note that $\{\mathcal{M}_\alpha\}_{\alpha\in I}$ is a totally ordered chain in $\mathcal{S}$. Set $\mathcal{M} = \left(\bigcup_{\alpha\in I} \mathcal{M}_\alpha\right)^-$. Since $\{\mathcal{M}_\alpha\}$ is a chain in $\mathcal{S}, \mathcal{M}$ is a subspace of $\mathcal{H}$ and $\mathcal{M}$ is invariant for T. To show that $T|\mathcal{M} \in \mathcal{F}$, let $T_\alpha \in \mathcal{L}(\mathcal{H})$ be defined by

$$T_\alpha = (T|\mathcal{M}_\alpha) \oplus (\lambda 1_{\mathcal{M}\ominus\mathcal{M}_\alpha}) .$$

Note that $T_\alpha \in \mathcal{F}$ and

$$T_\alpha h \longrightarrow Th \quad \text{for} \quad h \in \bigcup_{\alpha\in I} \mathcal{M}_\alpha . \tag{3.17}$$

Since $\sup\{||T_\alpha|| : \alpha \in I\} < \infty$, (3.17) implies that T_α converges to $T|\mathcal{M}$ in the strong operator topology. By Proposition 3.14, $T|\mathcal{M} \in \mathcal{F}$. This completes the proof of Theorem 3.16.

Now, if $\mathcal{F}$ is a family, $T \in \mathcal{F}$, K is a polynomially convex subset of $\mathbf{C}, \mathcal{F}_K$ is a family of operators and $\mathcal{M}$ is a maximal invariant subspace for T such that $T|\mathcal{M} \in \mathcal{F}_K$ (via Theorem 3.16), then one can investigate the connection between K, T and $\sigma(P_{\mathcal{M}^\perp}T|\mathcal{M}^\perp)$. The following proposition shows that if C is an exposed arc of $\sigma(T)$ satisfy the hypothesis of Proposition 1.22, then $\sigma(P_{\mathcal{M}^\perp}T|\mathcal{M}^\perp) \cap C = \phi$.

We begin with the following lemma.

Lemma 3.18. *If $T \in \mathcal{L}(\mathcal{H})$, K is compact, $\mathcal{M}$ is an invariant subspace for T and $\sigma(T|\mathcal{M}) \subseteq K$, then $\sigma(P_{\mathcal{M}^\perp}T|\mathcal{M}^\perp)\backslash K = \sigma(T)\backslash K$.*

Proof. Let T have the block operator form

$$T = \begin{bmatrix} T_0 & E \\ 0 & T_1 \end{bmatrix} \tag{3.19}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

Let $\lambda \notin K$. Since $\sigma(T_0) \subseteq K$, $T_0 - \lambda$ is invertible. By (3.19), $T^* - \overline{\lambda}$ has the block operator form

$$T^* - \overline{\lambda} = \begin{bmatrix} T_1^* - \overline{\lambda} & E^* \\ 0 & T_0^* - \overline{\lambda} \end{bmatrix} \tag{3.20}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M}^{\perp} \oplus \mathcal{M}$. By Problem 71 of [Ha] and (3.20), $T^* - \overline{\lambda}$ is invertible if and only if $T_1^* - \overline{\lambda}$ is invertible. Thus, $\sigma(T_1)\backslash K = \sigma(T)\backslash K$. This completes the proof of Lemma 3.18.

Proposition 3.21. *Let $\mathcal{F}$ be a family of operators, $T \in \mathcal{F} \cap \mathcal{L}(\mathcal{H})$ and K be a polynomially convex subset of* **C**. *Suppose that $\mathcal{F}_K$ is a family of operators and C is a subset of K such that C is an exposed arc of $\sigma(T) \cup K$, $\sigma(T) \cap C \neq \phi$ and for every connected component X of $\sigma(T) \cap C$, X is exposed to some order from both sides for T. If $\mathcal{M}$ is a subspace of $\mathcal{H}$ maximal with respect to the conditions that $\mathcal{M}$ is invariant for T and $T|\mathcal{M} \in \mathcal{F}_K$, then $\sigma(P_{\mathcal{M}^{\perp}}T|\mathcal{M}^{\perp})\backslash K = \sigma(T)\backslash K$ and $\sigma(P_{\mathcal{M}^{\perp}}T|\mathcal{M}^{\perp}) \cap C = \phi$.*

Proof. The first claim follows from Lemma 3.18.

If $\mathcal{M} = \mathcal{H}$, then the second statement holds trivially. Suppose that $\mathcal{M} \neq \mathcal{H}$ and $\sigma(P_{\mathcal{M}^{\perp}}T|\mathcal{M}^{\perp}) \cap C \neq \phi$. Let T have the block operator form

$$T = \begin{bmatrix} T_0 & E \\ 0 & T_1 \end{bmatrix} \tag{3.22}$$

with respect to $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Let $\mathcal{D}$ be an open disk such that

$$\mathcal{D} \cap (\sigma(T) \cup K) = C\ . \tag{3.23}$$

By Lemma 3.18,

$$\mathcal{D} \cap (\sigma(T) \cup K) = \mathcal{D} \cap (\sigma(T)\backslash K \cup K) = \mathcal{D} \cap (\sigma(T_1)\backslash K \cup K) = \mathcal{D} \cap (\sigma(T_1) \cup K).$$

Therefore, C is an exposed arc of $\sigma(T_1) \cup K$.

For $\lambda \in \mathcal{D}\backslash C$, $\lambda \notin K$ since $K \cap \mathcal{D} \subseteq (K \cup \sigma(T)) \cap C = C$. Therefore, by Lemma 3.18,

$$\|(T-\lambda)^{-1}\| = \left\| \begin{bmatrix} (T_0-\lambda)^{-1} & -(T_0-\lambda)^{-1}E(T_1-\lambda)^{-1} \\ 0 & (T_1-\lambda)^{-1} \end{bmatrix} \right\| \geq \|(T_1-\lambda)^{-1}\|,$$

and so every connected component X of $\sigma(T_1) \cap C$ which is also an arc is exposed to some order from both sides for T_1, since each such X is contained in a connected component of $\sigma(T) \cap C$. Since $\sigma(P_{\mathcal{M}^\perp} T|\mathcal{M}^\perp) \cap C \neq \phi$, Corollary 1.25 applies and there exists a nonzero subspace $\mathcal{N}$ of $\mathcal{M}^\perp$ invariant for T_1 such that $\sigma(T_1|\mathcal{N}) \subseteq C^-$. But then $\mathcal{M} \oplus (\mathcal{N} \oplus \{0\})$ is invariant for T and $\sigma(T|(\mathcal{M} \oplus (\mathcal{N} \oplus \{0\}))) \subseteq \sigma(T_0) \cup \sigma(T_1|\mathcal{N}) \subseteq K$. Thus $T|(\mathcal{M} \oplus (\mathcal{N} \oplus \{0\})) \in \mathcal{F}_K$, contradicting the maximality of $\mathcal{M}$. Therefore, $\sigma(P_{\mathcal{M}^\perp} T|\mathcal{M}^\perp) \cap C = \phi$. This completes the proof of Proposition 3.21.

We now illustrate the use of the above results to solve the equation $p(T) = 0$ for T for a particular $p \in \mathbf{C}[x, y]$. A similar approach can be used to deduce the roots of other polynomials p. If $T \in \mathcal{L}(\mathcal{H})$, then $\sigma_c(T)$ will denote the compression spectrum of T (i.e., λ such that $\bar{\lambda}$ is an eigenvalue of T^*).

Example 3.24 Let $r < s < t$ and $p(x, y) = \left(\frac{x-y}{2i} - r\right)\left(\frac{x-y}{2i} - s\right)\left(\frac{x-y}{2i} - t\right)$. If $T \in \mathcal{L}(\mathcal{H})$ and $p(T) = 0$, then, by Proposition 1.12, the approximate point spectrum of T is a subset of

$$\{\lambda \in \mathbf{C} : Im(\lambda) \in \{r, s, t\}\} . \tag{3.25}$$

Since every compact subset of the set of (3.25) is polynomially convex, the spectrum of T lies in the set of (3.25). By Proposition 2.6, if $\{Im(\lambda) : \lambda \in \sigma(T)\} = \{r\}$ ($\{s\}$, $\{t\}$, resp.), then $T - ri$ ($T - si$, $T - ti$, resp.) is self-adjoint. Therefore, by the Riesz Decomposition Theorem and Lemma 1 of [R-R] there exist subspaces $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{H}$ such that $\mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{H}$ and T has the block operator for

$$T = \begin{bmatrix} A + si & E_1 & E_2 \\ 0 & A_1 + ri & F_1 \\ 0 & 0 & A_2 + ti \end{bmatrix} \tag{3.26}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$ where A_0

is self-adjoint, A_1 is similar to a self-adjoint operator and A_2 is similar to a self-adjoint operator. Note that $\mathcal{M}_0, \mathcal{M}_1$ or $\mathcal{M}_2$ may be $\{0\}$. In fact, $\mathcal{M}_0 = \{0\}$ if and only if $\sigma(T) \cap (\mathbf{R} + it) = \phi$ and in this case the block decomposition of (3.26) is equivalent to this block matrix with the first row and column removed. Analogous relations between $\mathcal{M}_1 = \{0\}$, $\sigma(T) \cap (\mathbf{R} + ir) = \phi$ and the removal of the second row and column holds (as well as a third set of relations). These "special" cases do not change any of the arguments of this paper. For example, if we conclude $E_1 = 0$ somehow, this is also true if $\mathcal{M}_0 = \{0\}$ or $\mathcal{M}_1 = \{0\}$ as in these cases $\mathcal{L}(\mathcal{M}_1, \mathcal{M}_0) = \{0\}$.

By applying Proposition 2.17 to T with $a = 1, b = 0$ and a complimentary pair of invariant subspaces $\{\mathcal{M}_0, \mathcal{N}_0\}$ ($\mathcal{N}_0$'s existence is guaranteed by the Riesz Decomposition Theorem) for T implies that $\mathcal{M}_0$ reduces T, $E_1 = 0$ and $E_2 = 0$. Thus A_1 is self-adjoint.

The complete solution of the equation will be finished if we can fully understand the lower right 2 by 2 sub-block matrix of (3.26). There are two different cases.

As a first case, note that if $s - r \neq t - s$, then an application of Proposition 2.17 to $T|(\{0\} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2)$ with $a = 1, b = 0$ and a complimentary pair of invariant subspaces $\{\mathcal{M}_1, \mathcal{N}_1\}$ for $T|(\{0\} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2)$ ($\mathcal{M}_1 + \mathcal{N}_1 = \{0\} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$) yields $F_1 = 0$ and so T is normal.

The remaining case is when $s - r = t - s$. To motivate the following argument, note that if $u_1 \in \mathbf{R}$, $u_2 \in \mathbf{R}$ and $u_1 \neq u_2$, then the spaces $\ker(T - (u_1 + ir))$ and $\ker(T - (u_2 + it))$ are orthogonal to each other (to prove this, consider T restricted to $\operatorname{span}\{h, k\}$ with $h \in \ker(T - (u_1 + ir))$ and $k \in \ker(T - (u_2 + it))$ and then apply Corollary 2.17). Therefore, if the spectrum of T happened to have finitely many points, then (via the Riesz Decomposition Theorem and the appropriate shuffling of Hilbert spaces)

$$T \cong N \oplus \begin{bmatrix} A + ir & E \\ 0 & A + it \end{bmatrix} \tag{3.27}$$

where N is normal, A is self-adjoint and E commutes with A. In fact, (3.27) describes the complete solution. The proof which we now give uses Corollary 2.28, Theorem 2.36 and Theorem 3.16.

Let $X = \{x \in \mathbf{R} : \sigma(T) \cap \{x + ir, x + it\} \neq \phi\}$. We now claim that if X is not a singleton set, then $\{T, T^*\}''$ is not a factor. Since all solutions to $p(T) = 0$ for $\sigma(T) \subseteq \{u + ir, u + it\}$ for $u \in \mathbf{R}$ have the form of (3.27), it will follow from Theorem 2.36 that T has the form of (3.27).

If X is not connected, let σ_1 and σ_2 be compact subsets of $\mathbf{R}$ such that $\sigma_1 \cup \sigma_2 = X$. Let $K_j = \{\lambda \in \sigma(T) : Re(\lambda) \in \sigma_j\}$ for $j = 1, 2$. Note that K_1 and K_2 are compact, $K_1 \cup K_2 = \sigma(T)$ and $K_1 \cap K_2 = \phi$. By the Riesz Decomposition Theorem, there exist a complimentary pair $\{\mathcal{M}, \mathcal{N}\}$ of invariant subspaces for T such that $\sigma(T|\mathcal{M}) \subseteq K_1$ and $\sigma(T|\mathcal{N}) \subseteq K_2$. By Corollary 2.17 (with $a = 1, b = 0$), $\mathcal{M}$ reduces T. Let $A = T|\mathcal{M}$ and $B = T|\mathcal{M}^\perp$. By Corollary 2.28 (with $\mathcal{N}_k = \mathcal{M}^\perp$ for each $k \geq 1$), (2.31) holds and so $\{T, T^*\}''$ is not a factor.

We suppose for the remainder of the proof that X is connected. Let $\mathcal{F}$ be the collection of operators

$$\mathcal{F} = \{R : p(R) = 0 \quad \text{and} \quad ||R|| \leq ||T||\} .$$

$\mathcal{F}$ is family of operators. We claim that $\mathcal{F}_K$ is a family for every compact subset of $(\mathbf{R} + ri) \cup (\mathbf{R} + ti)$. Let K be such a compact set. K is polynomially convex and $R \in \mathcal{F}_K$. By the Riesz Decomposition Theorem, there exists a complimentary pair $\{\mathcal{M}_r, \mathcal{M}_t\}$ of invariant subspaces for R such that $\sigma(R|\mathcal{M}_r) \subseteq \mathbf{R} + ir$ and

$\sigma(R|\mathcal{M}_t) \subseteq \mathbf{R} + it$. Therefore, $(R - ir)|\mathcal{M}_r$ and $(R - it)|\mathcal{M}_t$ are self-adjoint and, by Lemma 1 of [R-R], R has the block operator form

$$R = \begin{bmatrix} A_0 + ir & E \\ 0 & A_1 + it \end{bmatrix}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M}_r \oplus \mathcal{M}_r^\perp$ for some $A_0 \in \mathcal{L}(\mathcal{M}_r)$, A_0 self-adjoint, $E \in \mathcal{L}(\mathcal{M}_r^\perp, \mathcal{M}_r)$ and $A_1 \in \mathcal{L}(\mathcal{M}_r^\perp)$ such that A_1 is similar to a self-adjoint operator. Let $p_0(x,y) = (\frac{x-y}{2i} - r)(\frac{x-y}{2i} - s)$. Since $p_0 = p_0^\vee$ and $p_0(A_0 + ir) = 0$, $p_0(R)$ is self-adjoint and $p_0(R)$ has the block operator form

$$p_0(R) = \begin{bmatrix} 0 & W \\ W^* & Z \end{bmatrix}$$

for some $W \in \mathcal{L}(\mathcal{M}_r^\perp, \mathcal{M}_r)$ and some self-adjoint operator $Z \in \mathcal{L}(\mathcal{M}_r^\perp)$. Now, since $p(R) = 0$, we see that

$$\frac{1}{2i}\begin{bmatrix} 0 & W \\ W^* & Z \end{bmatrix} R - \frac{1}{2i}R^* \begin{bmatrix} 0 & W \\ W^* & Z \end{bmatrix} - t \begin{bmatrix} 0 & W \\ W^* & Z \end{bmatrix} = 0\,.$$

Therefore,

$$\frac{1}{2i}W^*(A_0 + ir) - \frac{1}{2i}(A_1^* - it)W - tW^* = 0.$$

An application of Rosenblum's Theorem to this expression yields that $W = 0$. Now, computing $p_0(R)$ explicitly yields $A_0E = EA_1$ and $Z = \frac{1}{2}\,|\,E\,|^2 + p_0(A_1 + it)$. Computing $p(R)$ using the formulas for $p_0(R)$ yields

$$(\frac{1}{2}\,|\,E\,|^2 + p_0(A_1 + it))A_1$$

is self-adjoint. Since $A_0E = EA_1$, $|\,E\,|^2\, A_1 = E^*A_0E$ and $|\,E\,|^2\, A_1$ is self-adjoint. Therefore, $p(A_1 + it) = 0$ and so A_1 is self-adjoint. Now, for $\lambda \notin K$, using the facts that A_0 and A_1 are self-adjoint, $\sigma(A_0 + ir) \subseteq K$, $\sigma(A_1 + ic) \subseteq K$ and $||E|| \leq ||T||$

$$||(R - \lambda)^{-1}|| \;=\; \left\| \begin{bmatrix} A_0 + ir - \lambda & E \\ 0 & A_1 + it - \lambda \end{bmatrix}^{-1} \right\|$$

$$\begin{aligned}
&= \left\| \begin{bmatrix} (A_0+ir-\lambda)^{-1} & -(A_0+ir-\lambda)^{-1}E(A_1+it-\lambda)^{-1} \\ 0 & (A_1+it-\lambda)^{-1} \end{bmatrix} \right\| \\
&\le ||(A_0+ir-\lambda)^{-1}|| + ||(A_1+it-\lambda)^{-1}|| \\
&\quad + ||(A_0+ir-\lambda)^{-1}|| \, ||E|| \; ||(A_1+it-\lambda)^{-1}|| \\
&\le \frac{1}{\text{dist}(\lambda-ir,\sigma(A_0))} + \frac{1}{\text{dist}(\lambda-it,\sigma(A_1))} \\
&\quad + \frac{||T||}{\text{dist}(\lambda-ir,\sigma(A_0))\text{dist}(\lambda-it,\sigma(A_1))} .
\end{aligned}$$

Therefore, $\mathcal{F}_K$ is a family of operators by Proposition 3.12.

Let $u_1, u_3 \in \mathbf{R}$ be such that $X = [u_1, u_3]$ and $\mathcal{M}_r$ and $\mathcal{M}_t$ be the complimentary pair of invariant subspaces for T as described above. Since $T|\mathcal{M}_r$ and $T|\mathcal{M}_t$ are self-adjoint and $\mathcal{H}$ is separable, there exists $u_2 \in (u_1, u_3)$ such that $u_2 + ir \notin \sigma_c(T|\mathcal{M}_r)$ and $u_2 + it \notin \sigma_c(T|\mathcal{M}_t)$. Now, for $n \geq 1$, let

$$K_n = \{x+iy : x \in \left[u_1, u_2 - \frac{u_2-u_1}{n}\right], y \in \{r,t\}\}$$

and

$$L = \{x+iy : x \in [u_2,u_3], y \in \{r,t\}\} .$$

L is compact, K_n is compact for each $n \geq 1$ and $K_n \cap L = \phi$ for each $n \geq 1$. By Theorem 3.16, let $\mathcal{J}_n$ be subspaces of $\mathcal{H}$ maximal such that $\mathcal{J}_n$ is invariant for T, $T|\mathcal{J}_n \in \mathcal{F}_{L\cup K_n}$ and $\mathcal{J}_{n+1} \supseteq \mathcal{J}_n$. Let $\mathcal{N}_n$ and $\mathcal{M}_n$ be subspaces of $\mathcal{J}_n$ invariant for $T|\mathcal{J}_n$ such that $\sigma(T|\mathcal{N}_n) \subseteq K_n$ and $\sigma(T|\mathcal{M}_n) \subseteq L$. By the Riesz Decomposition Theorem, the maximality of $\mathcal{J}_n$ and the fact that $K_n \cap L = \phi$, $\mathcal{M}_n = \mathcal{M}_1$ and $\mathcal{N}_n \subseteq \mathcal{N}_{n+1}$ for each $n \geq 1$. By Corollary 2.17 (with $a = 1$ and $b = 0$), $\mathcal{M}_1$ is orthogonal to $\mathcal{N}_n$ for each n. Let $\mathcal{N}_\infty = \left(\bigcup_{n=1}^{\infty} \mathcal{N}_n\right)^-$ and note that $\mathcal{M}_1$ is orthogonal to $\mathcal{N}_\infty$.

By Proposition 3.18, $\sigma(P_{\mathcal{J}_n^\perp}T|\mathcal{J}_n^\perp)$ is a subset of

$$\left\{x+iy : x \in \{u_1,u_2,u_3\} \cup \left(u_2 - \frac{u_2-u_1}{n}, u_2\right) \text{ and } y \in \{r,t\}\right\} .$$

The Riesz Decomposition Theorem and the maximality of $\mathcal{J}_n$ implies that $\sigma(P_{\mathcal{J}_n^\perp}T|\mathcal{J}_n^\perp) \cap \{t_1 + ia, t_2 + ic, t_3 + ia, t_4 + ic\} = \phi$. Now, since $\mathcal{J}$ is invariant for T,

$$\sigma(T^*|\mathcal{J}_n^\perp) = \overline{\sigma(P_{\mathcal{J}_n^\perp}T|\mathcal{J}_n^\perp)}$$

and so the facts proven in this paragraph show that

$$\begin{aligned} \sigma(P_{\mathcal{H}\ominus(\mathcal{M}_1\oplus\mathcal{N}_\infty)}T|\mathcal{H}\ominus(\mathcal{M}_1\oplus\mathcal{N}_\infty)) &= \overline{\sigma(T^*|\mathcal{H}\ominus(\mathcal{M}_1\oplus\mathcal{N}_\infty))} \\ &\subseteq \{u_2 + ir, u_2 + it\}\,. \end{aligned}$$

We now claim that if $\mathcal{M}_1 \oplus \mathcal{N}_\infty \neq \mathcal{H}$, then either $u_2 + ir \in \sigma_c(T)$ or $u_2 + it \in \sigma_c(T)$. Indeed, if $\mathcal{M} \oplus \mathcal{N}_\infty \neq \mathcal{H}$, then $\sigma(P_{\mathcal{H}\ominus(\mathcal{M}_1\oplus\mathcal{N}_\infty)}T|\mathcal{H}\ominus(\mathcal{M}_1\oplus\mathcal{N}_\infty))$ is not empty. Suppose that $u_2 + it$ is in this set. But then there would exist a nontrivial subspace $\mathcal{W}$ invariant for T^* such that $\sigma(T^* \mid \mathcal{W}) = \{u_2 - it\}$. It is clear that $\mathcal{W} \subseteq \mathcal{M}_r^\perp$. Let S be an invertible operator such that SA_2S^{-1} is self-adjoint. Therefore,

$$\begin{aligned} T^* \mid \mathcal{W} &= (A_2 + it)^* \mid \mathcal{W} \\ &= S^*(SA_2S^{-1} + it)^*S^{-1^*} \mid \mathcal{W} \\ &= X_2((SA_2S^{-1} + it)^* \mid S^{-1^*}\mathcal{W})X_1 \end{aligned}$$

where $X_1 : \mathcal{W} \to S^{-1^*}\mathcal{W}$ is defined by $X_1 w = S^{-1^*}w$ for $w \in \mathcal{W}$ and $X_2 : S^{-1^*}\mathcal{W} \to \mathcal{W}$ is defined by $X_2 S^{-1^*}w = w$ for $w \in \mathcal{W}$. Since X_1 and X_2 are invertible, $(SA_2S^{-1} + it)^* \mid S^{-1^*}\mathcal{W}$ has the single point $u_2 - it$ in its spectrum. Since every subnormal operator whose spectrum is a single point is a scalar multiple of the identity, we may now conclude that $(SA_2S^{-1} + it)^*$ has $u_2 - it$ as an eigenvalue. Since $(A_2 + it)^*$ is similar to $(SA_2S^{-1} + it)^*$, $u_2 - it$ is an eigenvalue of $(A_2 + it)^*$. But then the above displayed equation implies that $u_2 - it$ is an eigenvalue of T^*

which was what was to be shown. An analogous argument holds for $u_2 + ir \in \sigma(P_{\mathcal{H}\ominus(\mathcal{M}_1\oplus\mathcal{N}_\infty)}T|\mathcal{H}\ominus(\mathcal{M}_1\oplus\mathcal{N}_\infty))$.

Since either $u_2 + ir \in \sigma_c(T)$ or $u_2 + it \in \sigma_c(T)$, $u_2 + ir$ or $u_2 + it$ would be in the compression spectrum of $T|\mathcal{M}_r$ or $T|\mathcal{M}_t$, contrary to the choice of u_2. Therefore, $\mathcal{M}_1 \oplus \mathcal{N}_\infty = \mathcal{H}$.

An application of Corollary 2.28 (with $A = T|\mathcal{M}_1$ and $B = T|\mathcal{N}_\infty$) shows that $\{T, T^*\}''$ is not a factor. Thus, (3.27) holds for some normal operator N, some self-adjoint A and some E which commutes with A (see the paragraph following (3.27)). This completes the classification of T when $p(T) = 0$.

The technique of Example 3.24 does *not* solve the equation $p(T) = 0$ for many polynomials p. For instance, the solution to the equation $(y - x)^3(T) = 0$ is given in [Ag1] via a model involving distributions. In contrast, the above techniques seem to only yield the following result whereas "most" of the solutions of this equation have compression spectrum which is an interval.

Example 3.28 Using techniques similar to the last example, it is easy to show that if $T \in \mathcal{L}(\mathcal{H})$, $n \geq 1$ and $(y - x)^n(T) = 0$, then

$$T \cong \int_\oplus T_\zeta d\nu(\zeta) \in \mathcal{L}(\int_\oplus \mathcal{H}_\zeta d\nu(\zeta))$$

for some measure ν and operators T_ζ such that $(y - x)^n(T_\zeta) = 0$ for almost every ζ and for almost every ζ and every $\lambda \in \sigma(T_\zeta)$, there exists a subspace $\mathcal{M}$ of $\mathcal{H}_\zeta$ invariant for T_ζ such that $\sigma(P_{\mathcal{M}^\perp}T_\zeta \mid \mathcal{M}^\perp) = \{\lambda\}$.

The techniques of Example 3.24 are less useful in solving $p(T) = 0$ when $\sigma(T)$ has non-empty interior. Chapters 4 and 5 will be devoted to one such example. Other examples are studied in [Ag-S] and [Ag8]. One large class of roots, whose spectrum has nonempty interior, which are easily classified, are subnormal roots. The following proposition shows that these are exactly the ones which arise from restrictions of normal roots.

Proposition 3.29. *Let $T \in \mathcal{L}(\mathcal{H})$, T be subnormal, $p \in \mathbf{C}[x,y]$ and $p(T) = 0$. If N is the minimal normal extension of T, then $p(N) = 0$.*

Proof. By Proposition I.2.3 of [LHypo], there exists a positive $\mathcal{L}(\mathcal{H})$-valued measure dQ on $\mathbf{C}$, with compact support, such that

$$S^{*n}S^m = \int_{\mathbf{C}} \overline{z}^n z^m dQ(z) . \tag{3.30}$$

Let $N \in \mathcal{L}(\mathcal{K})$ be the minimal normal extension of T and dE be its spectral measure. Since N is normal, $p(N) = 0$ if and only if $\sigma(N) \subseteq \Lambda_p$. This occurs if and only if the support of E lies in Λ_p. Since N is the minimal normal extension of T, the support of E lies in the support of Q, [C2]. Therefore, the proposition will follow if it can be shown that the support of Q lies in Λ_p.

Let $q(x,y) \in \mathbf{C}[x,y]$ be given by

$$q(x,y) = p(x,y)p^{\vee}(x,y) .$$

Computation using (3.30) and (1.11) yields

$$\begin{aligned} \int |p(\lambda,\overline{\lambda})|^2 dQ &= \int q(\lambda,\overline{\lambda}) dQ \\ &= q(T) \\ &= (p^{\vee}p)(T) \\ &= 0 . \end{aligned}$$

Since Q is a *positive* operator measure, this computation shows that the support of Q lies in Λ_p and completes the proof of Proposition 3.29.

Chapter 4

The Elementary Operator Theory of Isosymmetries

In this chapter, we introduce a class of operators termed isosymmetries and give their elementary properties.

Definition 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$. T is an* <u>*isosymmetry*</u> *if $T^{*2}T - T^*T^2 - T^* + T = 0$.*

Using (1.1), we see that T is an isosymmetry if and only if $((yx-1)(y-x))(T) = 0$. Therefore, Proposition 1.10 implies that every self-adjoint operator (i.e., T such that $(y-x)(T) = 0$) and every isometry (i.e., T such that $(yx-1)(T) = 0$) is an isosymmetry. We shall see that there are many other operators (e.g., nonnormal matrices) which are isosymmetries.

Chapter 1 yields the following two propositions which describe the spectral picture of an isosymmetry and the resolvent inequalities associated with an isosymmetry.

Proposition 4.2. *Let $T \in \mathcal{L}(\mathcal{H})$ and T be an isosymmetry. The approximate point spectrum of T is a subset of $\mathbf{R} \cup \partial\mathbf{D}$. The spectrum of T has the form $K \cup Y$ where K is a compact subset of $\mathbf{R} \cup \partial\mathbf{D}$ and Y is either $\mathbf{D}^-$, $\mathbf{D}^- \cap (LHP)^-$, $\mathbf{D}^- \cap (UHP)^-$ or the empty set where LHP is the lower half plane, $\{\lambda \in \mathbf{C}$:*

$Im\,\lambda < 0\}$, and UHP is the upper half plane, $\{\lambda \in \mathbf{C} : Im\,\lambda > 0\}$. If T is finitely cyclic, then the essential spectrum of T is a compact of $\mathbf{R} \cup \partial\mathbf{D}$.

Proof. If $\lambda \in \mathbf{C}$ and $(\bar{\lambda}\lambda - 1)(\bar{\lambda} - \lambda) = 0$, then $\lambda \in \mathbf{R} \cup \partial\mathbf{D}$. Therefore, by Proposition 1.12, the approximate point spectrum is a subset of $\mathbf{R} \cup \partial\mathbf{D}$. Since $\partial\sigma(T) \subseteq \sigma_{ap}(T) \subseteq \mathbf{R} \cup \partial\mathbf{D}$, $\sigma(T)$ must have the form $K \cup Y$ for some K and Y as stated in the hypothesis. If T is finitely cyclic, then $\sigma_e(T) \subseteq \mathbf{R} \cup \mathbf{D}^-$ by Proposition 1.12. This completes the proof of Proposition 4.2.

The following proposition follows from Propositions 1.15 and 1.17 and trivial algebraic manipulations.

Proposition 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$, T be an isosymmetry and K be a compact subset of $\mathbf{C}$. The following holds.*

(a) There exists $M > 0$ such that, whenever $\lambda \in K$, $\lambda \notin \sigma(T)$ and $\lambda \notin \mathbf{R} \cup \partial\mathbf{D}$,

$$\| (T-\lambda)^{-1} \| \leq \frac{M}{(|\lambda|^2 - 1)(Im\,\lambda)}. \tag{4.4}$$

In addition, if $\lambda_0 \in \mathbf{C}$ and K is a subset of the unbounded sector

$$S = \{\lambda_0 + re^{i\theta} : r \geq 0 \text{ and } \theta_0 \leq \theta \leq \theta_1\}$$

where $\theta_0 \in \mathbf{R}$ and $\theta_1 \in \mathbf{R}$, then the following hold.

(b) If $\lambda_0 \in \mathbf{R}$, $\lambda_0 \neq 1$, $\lambda_0 \neq -1$ and $S \cap \mathbf{R} = \{\lambda_0\}$, then there exist $C > 0$ and $\epsilon > 0$ such that, whenever $\lambda \in K$, $\lambda \notin \sigma(T)$ and $0 < |\lambda - \lambda_0| < \epsilon$, then

$$\| (\lambda - T)^{-1} \| \leq \frac{C}{|\lambda - \lambda_0|}. \tag{4.5}$$

(c) If $\lambda_0 \in \partial\mathbf{D}$, $\lambda_0 \neq 1$, $\lambda_0 \neq -1$ and the intersection of S and the tangent line to the unit circle at λ_0 is $\{\lambda_0\}$, then there exist $C > 0$ and $\epsilon > 0$ such that, whenever $\lambda \in K$, $\lambda \notin \sigma(T)$ and $0 < |\lambda - \lambda_0| < \epsilon$, then (4.5) holds.

(d) If $\lambda_0 = 1$ *or* $\lambda_0 = -1$ *and both* $S \cap \mathbf{R} = \{\lambda_0\}$ *and* $S \cap \{z : Re(z) = \lambda_0\} = \{\lambda_0\}$, *then there exist* $C > 0$ *and* $\epsilon > 0$ *such that, whenever* $\lambda \in K$, $\lambda \notin \sigma(T)$, $0 <\mid \lambda - \lambda_0 \mid< \epsilon$, *then*

$$\| (\lambda - T)^{-1} \| \leq \frac{C}{\mid \lambda - \lambda_0 \mid^2}. \tag{4.6}$$

Before continuing, note that (4.5) *cannot* be used instead of (4.6) in Proposition 4.3(d) since the 2×2 matrix

$$T = \begin{bmatrix} \lambda_0 & b \\ 0 & \lambda_0 \end{bmatrix} \in \mathbf{C}^{2,2}$$

is an isosymmetry for $\lambda_0 \in \{-1, 1\}$ and $b \in \mathbf{C}$ and, for $b \neq 0$, this matrix satisfies (4.6) but not (4.5). Since $(yx - 1)(y - x)$ can be written as $y(yx - 1) - (yx - 1)x$ or $y(y - x)x - (y - x)$, the isometry equation can be viewed in two different ways as illustrated by the following lemma. In the following lemma and throughout the paper, Δ_T will denote $T^*T - 1$ and $Im(T)$ will denote $\frac{1}{2i}(T - T^*)$ for any operator T.

Lemma 4.7. *Let* $T \in \mathcal{L}(\mathcal{H})$. *The following are equivalent.*

(a) T *is an isosymmetry.*

(b) $T^* Im(T) T = Im(T)$.

(c) $\Delta_T T = T^* \Delta_T$.

We may now exploit Rosenblum's Theorem and Lemma 4.7 to classify some isosymmetries based on the location of their spectrum.

Corollary 4.8. *Let* $T \in \mathcal{L}(\mathcal{H})$ *and* T *be an isosymmetry. If* $\sigma(T) \subseteq \mathbf{D}$ *or* $\sigma(T) \cap \mathbf{D}^- = \phi$, *then* T *is self-adjoint. If* $\sigma(T)$ *is contained in the open upper or open lower half plane, then* T *is unitary.*

Proof. If $\sigma(T) \subseteq \mathbf{D}^-$ or $\sigma(T) \cap \mathbf{D}^- = \phi$, then $\lambda\bar{\mu} \neq 1$ for any λ, $\mu \in \sigma(T)$. Therefore, by the equivalence of (4.7a) and (4.7b), an application of Rosenblum's Theorem ((2.2), with $A = T^*$, $B = T$ and $p(x, y) = yx - 1$) yields that $Im(T) = 0$ and so T is self-adjoint. The second statement follows analogously. This completes the proof of Corollary 4.8.

Before continuing, we record the following lemma which follows from Definition 4.1.

Lemma 4.9. *Let $T \in \mathcal{L}(\mathcal{H})$. T is an isosymmetry if and only if $-T$ is an isosymmetry.*

If $T \in \mathcal{L}(\mathcal{H})$ is an isosymmetry, then Corollary 4.8 motivates the analysis of block operator forms of T with respect to a Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ where $\mathcal{M}$ is invariant for T and $T \mid \mathcal{M}$ is either self-adjoint or unitary. The following two propositions begin this analysis.

Lemma 4.10. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $N \in \mathcal{L}(\mathcal{H})$, $E \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $X \in \mathcal{L}(\mathcal{K})$. If N is self-adjoint or unitary, $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ and T has the block operator form*

$$T = \begin{bmatrix} N & E \\ 0 & X \end{bmatrix}, \tag{4.11}$$

then T is an isosymmetry if and only if

$$NEX = E \tag{4.12}$$

and

$$X \text{ is an isosymmetry.} \tag{4.13}$$

Proof. Using (4.11), we compute

$$\Delta_T T = \begin{bmatrix} \Delta_N N & \Delta_N E + N^* E X \\ E^* N^2 & E^* N E + E^* E X + \Delta_X X \end{bmatrix}. \tag{4.14}$$

By Lemma 4.7, T is an isosymmetry if and only if

$$\Delta_N N \text{ is self-adjoint,} \tag{4.15}$$

$$N^{*2}E = \Delta_N E + N^*EX, \tag{4.16}$$

and

$$E^*NE + E^*EX + \Delta_X X \text{ is self-adjoint.} \tag{4.17}$$

Suppose that N is unitary. $\Delta_N = 0$ and so (4.15) holds and (4.16) is equivalent to (4.12). If T is an isosymmetry, then (4.12) holds and by (4.12)

$$\begin{aligned} E^*NE + E^*EX &= X^*E^*N^*NE + E^*EX \\ &= X^*E^*E + E^*EX\,. \end{aligned} \tag{4.18}$$

Thus, (4.13) holds by (4.17), (4.18) and Lemma 4.7. On the other hand, suppose (4.12) and (4.13) hold. By (4.12),

$$\begin{aligned} N^{*2}E - \Delta_N E - N^*EX &= N^{-2}E - N^{-1}EX \\ &= N^{-2}(E - NEX) \\ &= 0\,. \end{aligned}$$

Therefore, (4.16) holds. By (4.12), (4.18) holds and $\Delta_X X$ is self-adjoint by Lemma 4.7 and (4.13). Thus (4.17) holds. In summary, we have shown that Lemma 4.10 holds for N unitary.

Suppose that N is self-adjoint. Since N is self-adjoint, $\Delta_N N = (N^2-1)N$ is self-adjoint and (4.15) holds. Since N is self-adjoint, (4.16) is equivalent to (4.12). Therefore, it remains to show that if N is self-adjoint and (4.12) holds, then (4.17) holds if and only if (4.13) holds. Now, E^*NE is self-adjoint and, by (4.12)

$$E^*EX = (NEX)^*EX = X^*E^*NEX$$

is self-adjoint. Therefore, by Lemma 4.7, (4.13) and (4.17) are equivalent. In summary, Lemma 4.10 holds for N self-adjoint. This completes the proof of Lemma 4.10.

Lemma 4.19. *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{J}$ be Hilbert spaces and $T \in \mathcal{L}(\mathcal{M} \oplus \mathcal{N} \oplus \mathcal{J})$ be given by*

$$T = \begin{bmatrix} N_1 & B & E_1 \\ 0 & N_2 & E_2 \\ 0 & 0 & X \end{bmatrix} \tag{4.20}$$

where $N_1 \in \mathcal{L}(\mathcal{M})$, $B \in \mathcal{L}(\mathcal{N}, \mathcal{M})$, $E_1 \in \mathcal{L}(\mathcal{J}, \mathcal{M})$, $N_2 \in \mathcal{L}(\mathcal{N})$, $E_2 \in \mathcal{L}(\mathcal{J}, \mathcal{N})$ and $X \in \mathcal{L}(\mathcal{J})$. If N_1 and N_2 are self-adjoint or unitary, then T is an isosymmetry if and only if

$$N_1 B N_2 = B, \tag{4.21}$$

$$\begin{bmatrix} N_1 & B \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} X = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{4.22}$$

and

$$X \text{ is an isosymmetry.} \tag{4.23}$$

Proof. By Lemma 4.10 (with $\mathcal{H} = \mathcal{M}$ and $\mathcal{K} = \mathcal{N} \oplus \mathcal{J}$), T is an isosymmetry if and only if

$$N_1(BE_2 + E_1 X) = E_1, \tag{4.24}$$

$$\begin{bmatrix} N_2 & E_2 \\ 0 & X \end{bmatrix} \text{ is an isosymmetry,} \tag{4.25}$$

and (4.21) hold.

By Lemma 4.10, (4.25) holds if and only if

$$N_2 E_2 X = E_2 \tag{4.26}$$

and (4.23) hold. Therefore, T is an isosymmetry if and only if (4.21), (4.23), (4.24) and (4.26) hold. Now, since (4.22) holding is equivalent to (4.24) and (4.26) holding, we see that T is an isosymmetry if and only if (4.21), (4.22) and (4.23) hold. This completes the proof of Lemma 4.19.

We now describe the block operator form of an isosymmetry such that $\ker(T^*T - 1) \neq \{0\}$.

Lemma 4.27. *Let $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{M} = \ker(T^*T - 1)$ and suppose that T is an isosymmetry. $\mathcal{M}$ is invariant for T. If $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{M}_0$ is invariant for T, then T has the block operator form*

$$T = \begin{bmatrix} V & E \\ 0 & X \end{bmatrix} \tag{4.28}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_0^\perp$ where $V \in \mathcal{L}(\mathcal{M}_0)$, $E \in \mathcal{L}(\mathcal{M}_0^\perp, \mathcal{M}_0)$, $X \in \mathcal{L}(\mathcal{M}_0^\perp)$,

$$V \text{ is an isometry,} \tag{4.29}$$

$$V^*E = 0 \tag{4.30}$$

and

$$(E^*E + X^*X - 1)X \text{ is self-adjoint.} \tag{4.31}$$

In addition, if $\mathcal{M}_0 = \mathcal{M}$, then $E^*E + X^*X - 1$ is injective.

Proof. Since T is an isosymmetry (4.7c) holds and so $\mathcal{M}$ is invariant for T. Now, if $\mathcal{M}_0$ is invariant for T, then T has the block operator form (4.28) with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_0^\perp$ for some operators $V \in \mathcal{L}(\mathcal{M}_0)$, $E \in \mathcal{L}(\mathcal{M}_0^\perp, \mathcal{M}_0)$ and $X \in \mathcal{L}(\mathcal{M}_0^\perp)$. Now,

$$T^*T - 1 = \begin{bmatrix} V^*V - 1 & V^*E \\ E^*V & E^*E + X^*X - 1 \end{bmatrix}. \tag{4.32}$$

Since $\mathcal{M}_0 \subseteq \ker(T^*T-1)$, $V^*V = 1$ and $V^*E = 0$. Therefore, (4.29) and (4.30) hold. Using (4.32) and (4.28), we compute that

$$(4.33) \qquad (T^*T-1)T = \begin{bmatrix} 0 & 0 \\ 0 & (E^*E + X^*X - 1)X \end{bmatrix}.$$

Since T is an isosymmetry, Lemma 4.7 implies that $(T^*T-1)T$ is self-adjoint and so (4.31) holds.

If $\mathcal{M}_0 = \mathcal{M}$, then $T^*T - 1 = 0 \oplus (E^*E + X^*X - 1) \in \mathcal{L}(\ker(T^*T-1) \oplus \operatorname{ran}(T^*T-1)^-)$ and so $E^*E + X^*X - 1$ is injective. This completes the proof of Lemma 4.27.

A connection between isosymmetries and a certain collection of unbounded linear transformations exists. For $T \in \mathcal{L}(\mathcal{H})$ such that $1 \notin \sigma_p(T)$, let $\mathcal{D}_T = \operatorname{ran}(T-1)^2$ and L_T be the (possibly unbounded) linear transformation from $\mathcal{D}_T$ into $\mathcal{L}(\mathcal{H})$ defined by

$$L_T(T-1)^2 h = (T+1)^2 h \text{ for } h \in \mathcal{H}.$$

Lemma 4.34. *Let $T \in \mathcal{L}(\mathcal{H})$ and suppose that $1 \notin \sigma_p(T)$. T is an isosymmetry if and only if ${L_T}^2$ is symmetric.*

Proof. For $h \in \mathcal{H}$,

$$\langle L_T^2(T-1)^2 h, (T-1)^2 h\rangle = \langle (T+1)^2 h, (T-1)^2 h\rangle = \langle (T^*-1)^2(T+1)^2 h, h\rangle.$$

Therefore, L_T^2 is symmetric if and only if $(T^*-1)^2(T+1)^2$ is self-adjoint. Now, since

$$(y-1)^2(x+1)^2 - (y+1)^2(x-1)^2 = 4(yx-1)(y-x),$$

Proposition 1.2 implies that

$$(T^*-1)^2(T+1)^2 - (T^*+1)^2(T-1)^2 = 4(yx-1)(y-x)(T).$$

Thus, L_T^2 is symmetric if and only if T is an isosymmetry. This completes the proof of Lemma 4.34.

An analogous phenomena occurs if T is an isosymmetry and $-1 \notin \sigma_p(T)$ (see Lemma 4.9).

Chapter 5

Classification of Several Classes of Isosymmetries

In this chapter, we classify those isosymmetries T such that either $\sigma(T) = \partial\sigma(T)$, T is a contraction (i.e., $T^*T \leq 1$), T is hyponormal, $Im(T) \geq 0$, $Im(T) \leq 0$ or $T^*T \geq 1$.

Recall from Proposition 4.2 that if T is an isosymmetry, then $\sigma(T) \subseteq \mathbf{R} \cup \partial\mathbf{D}$ and $\partial\sigma(T) \subseteq \sigma_{ap}(T) \subseteq \mathbf{R} \cup \partial\mathbf{D}$. Via a sequence of three lemmas, we shall prove Theorem 5.1 which gives a *formula* for any isosymmetry T such that $\sigma(T) = \partial\sigma(T)$ (i.e., $\sigma(T) \subseteq \mathbf{R} \cup \partial\mathbf{D}$).

Theorem 5.1 *If T is an isosymmetry and $\sigma(T) \subseteq \mathbf{R} \cup \partial\mathbf{D}$, then there exist Hilbert spaces $\mathcal{H}, \mathcal{K}, \mathcal{M}$, and $\mathcal{N}$ and operators*

$$A, B \in \mathcal{L}(\mathcal{H}), \quad U, C \in \mathcal{L}(\mathcal{K}), \quad A_0 \in \mathcal{L}(\mathcal{M}) \text{ and } U_0 \in \mathcal{L}(\mathcal{N})$$

such that A, A_0 are self-adjoint, U, U_0 are unitary, B commutes with A, C commutes with U and T is (unitarily equivalent to)

$$\text{(5.2)} \qquad \begin{bmatrix} A & B \\ 0 & A^{-1} \end{bmatrix} \oplus \begin{bmatrix} U & C \\ 0 & U^{-1} \end{bmatrix} \oplus A_0 \oplus U_0 \, .$$

B and C can be taken to be injective and positive. Any of the summands in the above expression may be absent. (That is, the corresponding Hilbert space may be $\{0\}$.)

We begin by proving Theorem 5.1 in the special case that either 1 or -1 is not in the spectrum of the isosymmetry.

Lemma 5.3 *If T is an isosymmetry and either 1 or -1 is not in the spectrum of T, then the conclusion of Theorem 5.1 holds for T.*

Proof. Suppose that $1 \notin \sigma(T)$. Set $R = (T+1)(T-1)^{-1}$. By Lemma 4.34, R^2 is symmetric. Since R is bounded, R^2 is self-adjoint and, by Corollary 3 of [R-R], there exist Hilbert spaces $\mathcal{H}, \mathcal{K}, \mathcal{M}$, and $\mathcal{N}$ and operators

$$B_1, C_1 \in \mathcal{L}(\mathcal{H}), \quad B_2, C_2 \in \mathcal{L}(\mathcal{K}), \quad D_1 \in \mathcal{L}(\mathcal{M}) \quad \text{and} \quad D_2 \in \mathcal{L}(\mathcal{N}),$$

such that the D_j are self-adjoint, the B_j are self-adjoint , the B_j and C_j are positive and injective and B_j commutes with C_j, for $j = 1, 2$, and R is unitarily equivalent to

$$\begin{bmatrix} B_1 & C_1 \\ 0 & -B_1 \end{bmatrix} \oplus \begin{bmatrix} iB_2 & C_2 \\ 0 & -iB_2 \end{bmatrix} \oplus D_1 \oplus iD_2 \tag{5.4}$$

By the spectral mapping theorem, $1 \notin \sigma(R)$ and neither 1 nor -1 is in $\sigma(B_1)$. If we set $A = (B_1+1)(B_1-1)^{-1}$, $A_0 = (D_1+1)(D_1-1)^{-1}$, $U = (iB_2+1)(iB_2-1)^{-1}$, $U_0 = (iD_2+1)(iD_2-1)^{-1}$, $B = 2C_1(B_1^2-1)^{-1}$ and $C = -2C_2(B_2^2+1)^{-1}$, then, since $T = (R+1)(R-1)^{-1}$, T is unitarily equivalent to (5.2). Since D_1, D_2, B_1 and B_2 are self-adjoint , A_0 and A are self-adjoint and U and U_0 are unitary. Since B_j commutes with C_j for $j = 1, 2$, A commutes with B and U commutes with C. To complete the proof in the case when $1 \notin \sigma(T)$, we need to show that B and C can be taken to be positive and injective. Note that since C_1 and C_2 are injective,

B and C are also. Since B_1, B_2, C_1 and C_2 are self-adjoint , B and C are also. If S_1 is the signum of B (i.e., $S_1 = P_{E((0,\infty))} - P_{E((-\infty,0))}$ where E is the spectral measure of B), then S_1 is unitary and

$$\begin{bmatrix} S_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 1 \end{bmatrix}^* = \begin{bmatrix} A & |B| \\ 0 & A^{-1} \end{bmatrix}.$$

Since $|B|$ commutes with A, we may take B to be positive. An analogous argument shows that C may be taken to be positive.

Now, if $-1 \notin \sigma(T)$, then $1 \notin \sigma(-T)$. Since $-T$ is an isosymmetry and $1 \notin \sigma(-T)$, the conclusion of Theorem 5.1 holds for $-T$ as shown in the first part of the proof. Let A, B, U, C, A_0 and U_0 be as in the conclusion with $B \geq 0$ and $C \geq 0$ for the isosymmetry $-T$. By conjugating the matrix of (5.2) for $-T$ by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \oplus 1 \oplus 1$$

we see that the conclusion of Theorem 5.1 (including the constraint on positivity) holds for T. This completes the proof of Lemma 5.3.

Lemma 5.5 *Suppose that $T \in \mathcal{L}(\mathcal{H})$, T is an isosymmetry and the conclusion of Theorem 5.1 holds for T. If $\mathcal{K}$ is a Hilbert space, $E \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{M})$, and $X \in \mathcal{L}(\mathcal{K})$, then the operator*

$$R = \begin{bmatrix} T & E \\ 0 & X \end{bmatrix}$$

is an isosymmetry if and only if $TEX = E$ and X is an isosymmetry.

Proof. Since the conclusion of Theorem 5.1 holds for T, R has the block operator form

$$(5.6) \qquad \begin{bmatrix} A & B & 0 & 0 & 0 & 0 & E_1 \\ 0 & A^{-1} & 0 & 0 & 0 & 0 & E_2 \\ 0 & 0 & U & C & 0 & 0 & E_3 \\ 0 & 0 & 0 & U^{-1} & 0 & 0 & E_4 \\ 0 & 0 & 0 & 0 & A_0 & 0 & E_5 \\ 0 & 0 & 0 & 0 & 0 & U_0 & E_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

acting on $\mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{K}$ where A, B, U, C, A_0 and U_0 have the properties listed in Theorem 5.1. By applying Lemma 4.10, Lemma 4.19 and the observation that

$$\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} Z = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

if and only if

$$X_1 \begin{bmatrix} 0 & E_1 \end{bmatrix} \begin{bmatrix} X_2 & E_2 \\ 0 & Z \end{bmatrix} = \begin{bmatrix} 0 & E_1 \end{bmatrix} \qquad \text{and}$$

$$X_2 E_2 Z = E_2 \,,$$

we see that R is an isosymmetry if and only if $TEX = E$ and X is an isosymmetry. This completes the proof of Lemma 5.5.

Lemma 5.7 *Let $\mathcal{F}$ be a family of operators and K be a compact subset of $\mathbf{R} \cup \partial\mathbf{D}$ such that either 1 or -1 is not a member of K. $\mathcal{F}_K$ is a family of operators.*

Proof. By Lemma 4.9, we may suppose that $1 \notin \sigma(T)$. Since $K \subseteq \mathbf{R} \cup \partial\mathbf{D}$, K is compact and $1 \notin \sigma(T)$, K is polynomially convex.

We claim that for every $\lambda \notin K$,

$$(5.8) \qquad \sup\{\|(R-\lambda)^{-1}\| : R \in \mathcal{F}\} < \infty \,.$$

Let $R \in \mathcal{F}$ and $\lambda \notin K$. R is unitarily equivalent to (5.2) for the appropriate choices of A, B, U, C, A_0 and U_0. Since $\lambda \notin \sigma(R)$, $\lambda \notin \sigma(A)$, $\frac{1}{\lambda} \notin \sigma(A)$, $\lambda \notin \sigma(U)$, $\frac{1}{\lambda} \notin \sigma(U)$,

$\lambda \notin \sigma(A_0)$ and $\lambda \notin \sigma(U_0)$. Therefore, $(R-\lambda)^{-1}$ is unitarily equivalent to the direct sum of $(A_0-\lambda)^{-1}$, $(U_0-\lambda)^{-1}$,

$$\begin{bmatrix} (A-\lambda)^{-1} & -(A-\lambda)^{-1}B(A^{-1}-\lambda)^{-1} \\ 0 & (A^{-1}-\lambda)^{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} (U-\lambda)^{-1} & -(U-\lambda)^{-1}C(U^{-1}-\lambda)^{-1} \\ 0 & (U^{-1}-\lambda)^{-1} \end{bmatrix}.$$

Therefore, since A, A_0, U and U_0 are normal and $\|B\|+\|C\| \leq 2\|R\| \leq 2\|T\|$,

$$\begin{aligned} \|(R-\lambda)^{-1}\| &\leq \frac{1}{dist(\lambda,K)} \\ &+ \|B\|\|(A-\lambda)^{-1}\|\|(A^{-1}-\lambda)^{-1}\| \\ &+ \|C\|\|(U-\lambda)^{-1}\|\|(U^{-1}-\lambda)^{-1}\| \\ &\leq \frac{1}{dist(\lambda,K)} + \frac{\|B\|+\|C\|}{dist(\lambda,K)^2} \\ &\leq \frac{1}{dist(\lambda,K)} + \frac{2\|T\|}{dist(\lambda,K)^2}. \end{aligned}$$

Thus, (5.8) holds. An application of Proposition 3.12 completes the proof of Lemma 5.7.

We now prove Theorem 5.1. Let RHP denote the right half plane: $RHP = \{\lambda : Re(\lambda) \geq 0\}$.

Proof of Theorem 5.1

Let $\mathcal{F}$ be the family of operators

$$\mathcal{F} = \{R : \|R\| \leq \|T\| \text{ and } R \text{ is an isosymmetry}\}.$$

Let K be the compact set

$$K = \{\lambda \in \mathbf{R} \cup \partial\mathbf{D} : -\|T\| \leq Re(\lambda) \leq 0\}.$$

By Lemma 5.7, $\mathcal{F}_K$ is a family of operators. Let $\mathcal{M}$ be the subspace of $\mathcal{H}$ which is maximal with respect to the conditions that $\mathcal{M}$ is invariant for T and $T \mid \mathcal{M} \in \mathcal{F}_K$.

If $\mathcal{M} = \mathcal{H}$, then Theorem 5.1 holds by Lemma 5.3. We shall assume for the rest of the proof that $\mathcal{M} \neq \mathcal{H}$. T has the block operator form

$$\begin{bmatrix} T_0 & W \\ 0 & T_1 \end{bmatrix}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.

We first claim that $\sigma(T_1) \subseteq (RHP)^{-} \cap (\mathbf{R} \cup \partial\mathbf{D})$. By Lemma 5.5, T_1 is an isosymmetry. Since both $\sigma(T)$ and $\sigma(T_0)$ are subsets of $\mathbf{R} \cup \partial\mathbf{D}$, $\sigma(T_1) \subseteq \mathbf{R} \cup \partial\mathbf{D}$. By Proposition 3.21, $\sigma(T_1) \backslash (RHP)^{-} \subseteq \{1\}$. If $1 \in \sigma(T_1)$, then 1 is an isolated point of $\sigma(T_1)$ and so by the Riesz Decomposition Theorem there exists an invariant subspace $\mathcal{N}$ of T_1 such that $\sigma(T_1 \mid \mathcal{N}) = \{1\}$. But then $\mathcal{M} \oplus \mathcal{N}$ would be invariant for T and the restriction of T to this subspace would be in $\mathcal{F}_K$. This would contradict the maximality of $\mathcal{M}$. Therefore, $1 \notin \sigma(T_1)$ and so $\sigma(T_1) \subseteq (RHP)^{-} \cap (\mathbf{R} \cup \partial\mathbf{D})$.

Since $\mathcal{M} \oplus (\ker(T-i) + \ker(T) + \ker(T+i))^{-}$ is invariant for T and T restricted to this invariant subspace is in $\mathcal{F}_K$, the maximality of $\mathcal{M}$ implies neither $i, 0$ nor $-i$ is a member of $\sigma_p(T_1)$.

We now claim that there exists a collection $\{\mathcal{K}_n\}_{n=1}^{\infty}$ of subspaces of $\mathcal{M}^{\perp}$ such that for each natural number n, $\mathcal{K}_n$ reduces T_1, $Re(\lambda) \geq \frac{\|T\|}{n+1}$ for each $\lambda \in \sigma(T_1 \mid \mathcal{K}_n)$ and

$$\mathcal{M}^{\perp} = \oplus_{n \geq 1} \mathcal{K}_n \, .$$

Since T_1 is an isosymmetry and $1 \notin \sigma(T_1)$, T_1 has a decomposition in the form of (5.2) for the appropriate choices of A, B, U, C, A_0 and U_0. Let T_2 be the operator of (5.2) and L be a Hilbert space isomorphism such that $LT_2 = T_1 L$. Let E_1, E_2, E_3 and E_4 be spectral measures of A, A_0, U and U_0, respectively. For each natural number n, let

$$\Delta_n = \{\lambda \in \mathbf{R} \cup \partial\mathbf{D} : \frac{\|T\|}{n+1} < Re(\lambda) \leq \frac{\|T\|}{n}\}$$

and note that $A \mid E_1(\Delta_n) \geq \frac{1}{n+1}$, $A_0 \mid E_2(\Delta_n) \geq \frac{1}{n+1}$, $Re(U) \mid E_3(\Delta_n) \geq \frac{1}{n+1}$ and $Re(U_0) \mid E_4(\Delta_n) \geq \frac{1}{n+1}$. By the Flugede-Putnam Theorem [R-R2], $E_1(\Delta_n)$ reduces B and $E_3(\Delta_n)$ reduces C. If we set

$$\mathcal{N}_n = E_1(\Delta_n) \oplus E_1(\Delta_n) \oplus E_2(\Delta_n) \oplus E_3(\Delta_n) \oplus E_3(\Delta_n) \oplus E_4(\Delta_n),$$

then $\mathcal{N}_n$ reduces T_2 and the span of the $\mathcal{N}_n$ is dense since neither i, 0 nor $-i$ is an eigenvalue of T_1 and $\sigma(T) \subset (RHP)^-$. Now, if $\mathcal{K}_n = L(\mathcal{N}_n)$, then $\mathcal{M}^\perp = \oplus_{n\geq 1}\mathcal{K}_n$ and for each n, $\mathcal{K}_n$ reduces T_1 and $Re(\lambda) \geq \frac{\|T\|}{n+1}$ for each $\lambda \in \sigma(T_1 \mid \mathcal{K}_n)$.

Since T is an isosymmetry and $\sigma(T_0) \subset K$, Lemma 5.5 implies that for each n,

$$T_0(WP_{\mathcal{K}_n})(T_1 \mid \mathcal{K}_n) = WP_{\mathcal{K}_n}. \tag{5.9}$$

Since $1 \notin \{ab : a \in \sigma(T_0) \text{ and } b \in \sigma(T_1 \mid \mathcal{K}_n)\}$, (5.9) and (2.2) imply that $WP_{\mathcal{K}_n} = 0$ for each n. Therefore, $W = 0$ and the conclusion of Theorem 5.1 holds for T, since it holds for both T_0 and T_1. This completes the proof of Theorem 5.1.

Lemma 5.10 *Let $T \in \mathcal{L}(\mathcal{H})$. If the spectrum of T is not contained in the unit disk and T is an isosymmetry, then there exists a nontrivial invariant subspace $\mathcal{M}$ for T, $A \in \mathcal{L}(\mathcal{M})$, $E \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{M})$, $X \in \mathcal{L}(\mathcal{M}^\perp)$ such that A is self-adjoint, X is an isosymmetry, $\sigma(X) \subseteq \mathbf{D}^-$ and T has the block operator form*

$$T = \begin{bmatrix} A & E \\ 0 & X \end{bmatrix}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$

Proof. Let $\mathcal{F}$ be the family of operators

$$\mathcal{F} = \{R : \|R\| \leq \|T\| \text{ and R is an isosymmetry}\}.$$

Let r be the spectral radius of T. By hypothesis, $r > 1$. Let $K = [-r, -1] \cup [1, r]$. K is polynomially convex and $\mathcal{F}_K$ is a family by Lemma 5.7. Let $\mathcal{N}$ be a subspace of $\mathcal{H}$ maximal with respect to the conditions that $\mathcal{N}$ is invariant for T and $T \mid \mathcal{N} \in \mathcal{F}_K$. Let

$$T = \begin{bmatrix} T_0 & F \\ 0 & T_1 \end{bmatrix}$$

be the block operator form of T with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$. By Proposition 3.21, $\sigma(T_1) \subset \mathbf{D}^- \cup \{-r, r\}$. If $\lambda \in \sigma(T_1) \backslash \mathbf{D}^-$, then, by the Riesz Decomposition Theorem, there exists a nonzero subspace $\mathcal{K}$ of $\mathcal{N}^\perp$ which is invariant for T_1 and $\sigma(T_1) = \{\lambda\}$. But then $\mathcal{N} \oplus \mathcal{K}$ would be invariant for T and the restriction of T to this subspace is a member of $\mathcal{F}_K$. This would contradict the maximality of $\mathcal{N}$. Therefore, $\sigma(T_1) \subseteq \mathbf{D}^-$. Now, by Theorem 5.1, T has the block operator decomposition

$$T = \begin{bmatrix} A & 0 & 0 & 0 & 0 & W_1 \\ 0 & 1 & B & 0 & 0 & W_2 \\ 0 & 0 & 1 & 0 & 0 & W_3 \\ 0 & 0 & 0 & -1 & C & W_4 \\ 0 & 0 & 0 & 0 & -1 & W_5 \\ 0 & 0 & 0 & 0 & 0 & W_6 \end{bmatrix}$$

with respect to a Hilbert space decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ where A is self-adjoint and B, C and the W's are operators on the appropriate spaces. By setting $\mathcal{N} = \mathcal{H}_0$,

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & W_1 \end{bmatrix}$$

and

$$X = \begin{bmatrix} 1 & B & 0 & 0 & W_2 \\ 0 & 1 & 0 & 0 & W_3 \\ 0 & 0 & -1 & C & W_4 \\ 0 & 0 & 0 & -1 & W_5 \\ 0 & 0 & 0 & 0 & W_6 \end{bmatrix}.$$

Lemma 4.10 implies that the conclusions of Lemma 5.10 hold. This completes the proof of Lemma 5.10.

We turn our attention to classifying isosymmetries which are contractions. We will give two different proofs of the result (Proposition 5.18). Both of these proofs will be based upon Theorem 5.1 and techniques involving Calkin algebras. The first proof requires a limit of contractions in the strong operator topology, gives an explicit lift for a finitely cyclic contractive isosymmetry and uses the Arveson Extension Theorem [Ag5]. The second proof relies on a characterization of subnormal contractions[Ag4] and the Arveson Extension Theorem [Ag5].

We first show that a finitely cyclic contractive isosymmetry is essentially normal. Let $\mathcal{K}(\mathcal{H})$ denote the norm closed ideal of compact operators in $\mathcal{L}(\mathcal{H})$.

Lemma 5.11 *If $T \in \mathcal{L}(\mathcal{H})$, T is finitely cyclic, T is an isosymmetry and T is a contraction, then T is essentially normal and $\sigma_e(T) \subseteq [-1,1] \cup \partial\mathbf{D}$.*

Proof. Let $K = [-1,1] \cup \partial\mathbf{D}$ and $\pi_0 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the natural map. Since $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a unital C^*-algebra, there exist a Hilbert space $\mathcal{M}$ and an injective unital *-representation $\pi : \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to \mathcal{L}(\mathcal{M})$. Since T is a finitely cyclic contractive isosymmetry, Proposition 1.12 implies that $\sigma_e(T) \subseteq K$ and so $\pi(\pi_0(T))$ is a contractive isosymmetry with spectrum in K. Thus, by Theorem 5.1, $\pi(\pi_0(T))$ is normal and so $\pi(\pi_0(T^*T - TT^*)) = 0$. Since π is injective, $\pi_0(T^*T - TT^*) = 0$ and T is essentially normal. This completes the proof of Lemma 5.11.

The following technical lemma will be invoked several times.

Lemma 5.12 *If $P \in \mathcal{L}(\mathcal{H})$, P is positive, $X \in \mathcal{L}(\mathcal{H})$ and $Re(X) \geq 0$, then PX does not have any negative eigenvalues.*

Proof. Let $t > 0$ and $h \in \mathcal{H}$ be such that $PXh = -th$. We wish to show that $h = 0$. Since P is positive,

$$\langle Xh, h\rangle = \frac{\langle Xh, PXh\rangle}{-t} \leq 0. \tag{5.13}$$

Since $Re(X) \geq 0$ and $\langle Xh, h\rangle \in \mathbf{R}$,

$$\langle Xh, h\rangle = Re\langle Xh, h\rangle = \langle Re(X)h, h\rangle \geq 0 .$$

Thus, by (5.13), $\langle PXh, Xh\rangle = 0$. Since P is positive, $PXh = 0$ and $h = -\frac{1}{t}PXh = 0$. This completes the proof of Lemma 5.12.

We now prove the following lemma which will play a key role in the first proof of the classification of isosymmetries which are contractions.

Lemma 5.14 *If $T \in \mathcal{L}(\mathcal{H})$, T is a finitely cyclic isosymmetry and T is a contraction, then $(T^*)^nT^n$ converges in the strong operator topology to a positive contraction P such that $T^*PT = P$,* $\ker(1-P) = \ker(1-T^*T)$*, $(1-P)(1-T) \geq 0$ and $(1-P)(1+T) \geq 0$.*

Proof. Since T is a contraction, $1 - T^*T \geq 0$ and

$$T^{*n}T^n - T^{*(n+1)}T^{n+1} = T^{*n}(1-T^*T)T^n \geq 0$$

for all $n \geq 1$. Thus,

$$1 \geq T^*T \geq T^{*2}T^2 \geq \dots \tag{5.15}$$

and $T^{*n}T^n$ converges in the strong operator topology to a positive contraction P ([Be],Proposition 1.1).

To see that $T^*PT = P$, note that for $h, k \in \mathcal{H}$,

$$\langle T^*PTh, k\rangle \;=\; \langle PTh, Tk\rangle$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \langle T^{*n}T^{(n+1)}h, Tk\rangle \\
&= \lim_{n\to\infty} \langle T^{*(n+1)}T^{n+1}h, k\rangle \\
&= \langle Ph, k\rangle \,.
\end{aligned}
$$

To see that $\ker(1-P) = \ker(1-T^*T)$, note that since T is an isosymmetry, Lemma 4.27 implies that T has the block operator form

$$
T = \begin{bmatrix} V & E \\ 0 & X \end{bmatrix} \tag{5.16}
$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ where $\mathcal{M} = \ker(1 - T^*T)$, $V \in \mathcal{L}(\mathcal{H})$, V is an isometry and $V^*E = 0$. Using the decomposition (5.16) yields

$$
P = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}.
$$

Thus, $\ker(1 - T^*T) = \mathcal{M} \subseteq \ker(1 - P)$. On the other hand, if $h \in \ker(1 - P)$, then (5.15) implies that

$$
\|h\|^2 = \langle Ph, h\rangle = \lim_{n\to\infty}\langle T^{*n}T^n h, h\rangle \le \langle T^*Th, h\rangle \le \|h\|^2.
$$

Therefore, $\|h\| = \|T^*Th\|$ and, since T is a contraction, $h \in \ker(1 - T^*T)$. Thus, $\ker(1 - P) = \ker(1 - T^*T)$.

Finally, let s be either 1 or -1. In order to show that $(1-P)(1-sT) \ge 0$, we will show that for all $n \ge 1$,

$$
(1 - T^{*n}T^n)(1 - sT) \ge 0 \,. \tag{5.17}
$$

First, note that

$$
\begin{aligned}
(1 - y^n x^n)(1 - sx) &= (1 - yx)\frac{1 - y^n x^n}{1 - yx}(1 - sx) \\
&= (1 - yx)\Big(\sum_{j=0}^{n-1} y^j x^j\Big)(1 - sx) \\
&= (1 - yx)\Big(\sum_{j=0}^{n-1} x^{2j}\Big)(1 - sx)
\end{aligned}
$$

modulo the ideal generated by $(yx-1)(y-x)$. Therefore, since T is an isosymmetry, Proposition 1.10 implies

$$(1-T^{*n}T^n)(1-sT) = (1-T^*T)(\sum_{j=0}^{n-1} T^{2j})(1-sT).$$

Thus, by Lemma 4.7, $(1-T^{*n}T^n)(1-sT)$ is self-adjoint .

Since T is an isosymmetry, a contraction and finitely cyclic, Lemma 5.11 implies that T is essentially normal and and $\sigma_e(T) \subseteq [-1,1] \cup \partial\mathbf{D}$. Since $(1-N^{*n}N^n)(1-sN) \geq 0$ for all normal operators N which are both contractions and isosymmetries, the same is true, by Proposition 3.29, for subnormal isosymmetries which are contractions. By [B-D-F], there exist $R \in \mathcal{L}(\mathcal{H})$ and $K \in \mathcal{L}(\mathcal{H})$ such that R is a contractive subnormal isosymmetry, K is compact and $T = R + K$. Therefore, $(1-T^{*n}T^n)(1-sT)$ is a compact perturbation of the positive operator $(1-R^{*n}R^n)(1-sR)$. By Weyl's Theorem ([R-R2], Theorem 0.10), if $(1-T^{*n}T^n)(1-sT)$ does not possess any negative eigenvalues, then (5.17) holds. Since T is a contraction, $1-T^{*n}T^n \geq 0$ and $Re(1-sT) \geq 0$. Therefore, by Lemma 5.14, $(1-T^{*n}T^n)(1-sT)$ does not possess any negative eigenvalues. This completes the proof of Lemma 5.14.

The following proposition classifies contractive isosymmetries. $T \in \mathcal{L}(\mathcal{H})$ will be called *cyclic* if there exists a vector γ such that the smallest invariant subspace for T containing γ is $\mathcal{H}$.

Proposition 5.18 *If $T \in \mathcal{L}(\mathcal{H})$, T is an isosymmetry, and T is a contraction, then T is unitarily equivalent to an operator of the form*

(5.19) $$(A \oplus U) \mid \mathcal{M}$$

where A is self-adjoint , $\|A\| \leq 1$ and U is unitary and $\mathcal{M}$ is an invariant subspace for $A \oplus U$.

Proof. Since an operator $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if, for each nonzero vector $\gamma \in \mathcal{H}$, T restricted to the smallest invariant subspace for T containing γ is subnormal [C2], it suffices to assume that T is cyclic. We will first construct a self-adjoint operator A and a unitary operator U and then give an intertwining isometry.

By Lemma 5.14, $T^{*n}T^n$ converges in the strong operator topology to a positive contraction P.

We first construct a Hilbert space $\mathcal{K}$ and $U \in \mathcal{L}(\mathcal{K})$. Let $\mathcal{K}_0 = \operatorname{ran}(P^{1/2})^-$. Since

$$\|P^{1/2}Th\|^2 = \langle T^*PTh, h\rangle = \langle Ph, h\rangle = \|P^{1/2}h\|^2,$$

there exist an isometry $V \in \mathcal{L}(\mathcal{K}_0)$ satisfying the relation

$$VP^{1/2}h = P^{1/2}Th, \qquad h \in \mathcal{H}.$$

By [F], there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{K}_0$ and a unitary operator $U \in \mathcal{L}(\mathcal{K})$ such that $\mathcal{K}_0$ is invariant for U and $U \mid \mathcal{K}_0 = V$.

Now, let $A_0 : \operatorname{ran}(1-P)^{1/2} \to \operatorname{ran}(1-P)^{1/2}$ be defined by

$$A_0(1-P)^{1/2}h = (1-P)^{1/2}Th, \qquad h \in \mathcal{H}.$$

A_0 is well-defined since $\ker(1-P)^{1/2} = \ker(1-P) = \ker(1-T^*T)$ is invariant for T by Lemma 4.27. The following two computations follow from Lemma 5.14 and prove that A_0 is symmetric and $\|A_0\| \leq 1$.

$$\begin{aligned} \|(1-P)^{1/2}h\|^2 - \langle A_0(1-P)^{1/2}h, (1-P)^{1/2}h\rangle &= \langle (1-P)h, h\rangle \\ &\quad -\langle (1-P)^{1/2}Th, (1-P)^{1/2}h\rangle \\ &= \langle (1-P)(1-T)h, h\rangle \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned}
\langle A_0(1-P)^{1/2}h,(1-P)^{1/2}h\rangle + \|(1-P)^{1/2}h\|^2 &= \langle (1-P)^{1/2}Th,(1-P)^{1/2}h\rangle \\
&\quad + \langle (1-P)h,h\rangle \\
&= \langle (1-P)(T+1)h,h\rangle \\
&\geq 0
\end{aligned}$$

Since $\|A_0\| \leq 1$ and A_0 is symmetric, A_0 extends by continuity to a self-adjoint contraction $A \in \mathcal{L}(\mathcal{J})$ where $\mathcal{J} = (\mathrm{ran}(1-P)^{1/2})^{-}$.

Finally, if $L : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{J} \oplus \mathcal{K})$ is defined densely by

$$L(h) = \begin{bmatrix} P^{1/2}h \\ (1-P)^{1/2}h \end{bmatrix},$$

then L is an isometry and

$$\begin{aligned}
\begin{bmatrix} U & 0 \\ 0 & A \end{bmatrix} Lh - LTh &= \begin{bmatrix} U & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} P^{1/2}h \\ (1-P)^{1/2}h \end{bmatrix} - \begin{bmatrix} P^{1/2}Th \\ (1-P)^{1/2}Th \end{bmatrix} \\
&= \begin{bmatrix} UP^{1/2}h - P^{1/2}Th \\ A(1-P)^{1/2}h - (1-P)^{1/2}Th \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

This completes the proof of Proposition 5.18.

We now give a second proof of Proposition 5.18.

Proof 2 of Proposition 5.18 . Let T be an isosymmetry which is a contraction. Proposition 5.18 will follow from Proposition 3.29 once it has been shown that T is subnormal. By [Ag4], since T is a contraction, T is subnormal if and only if for each $n \geq 1$,

(5.20) $$(1-yx)^n(T) \geq 0\,.$$

Now,

$$\begin{aligned}(1-yx)^{n+2} &= (1-yx)^n(1-yx)^2 \\ &= (1-yx)^n(1-x^2)^2 \\ &= (1-y^2)(1-yx)^n(1-x^2)\end{aligned}$$

modulo the ideal generated by $(yx-1)(y-x)$. Thus, by Proposition 1.2,

$$(1-yx)^{n+2}(T) = (1-T^{*2})((1-yx)^n(T))(1-T^2).$$

Thus, we need only show (5.20) for $n = 1$ and $n = 2$. Since T is a contraction, (5.20) holds for $n = 1$. Since $(1-yx)^2(T)$ is self-adjoint, we need only show that

$$\sigma((1-T^*T)(1-T^2)) \cap (-\infty, 0) = \phi. \tag{5.21}$$

Since $(1-N^*N)(1-N^2) \geq 0$ for any isosymmetry which is both subnormal and a contraction, Lemma 5.11 implies that $(1-T^*T)(1-T^2)$ is essentially positive. By Lemma 5.12, $(1-T^*T)(1-T^2)$ has no negative eigenvalues since $1-T^*T \geq 0$ and $Re(1-T^2) \geq 0$ (T is a contraction). Therefore, by Weyl's theorem, (5.20) holds for $n = 2$. This completes the second proof of Proposition 5.18.

The classification of contractive isosymmetries leads to the classification of hyponormal isosymmetries.

Proposition 5.22 *Let $T \in \mathcal{L}(\mathcal{H})$. T is both a isosymmetry and hyponormal if and only if T is unitarily equivalent to*

$$(A \oplus U) \mid \mathcal{M} \tag{5.23}$$

for some self-adjoint operator A, some unitary operator U and an invariant subspace $\mathcal{M}$ for $A \oplus U$.

Proof. By Lemma 5.10, there exists a Hilbert space $\mathcal{M}$ such that T has the block operator form

$$\begin{bmatrix} A & E \\ 0 & X \end{bmatrix}$$

where $A \in \mathcal{L}(\mathcal{M})$, A is self-adjoint, $E \in \mathcal{L}(\mathcal{M}^{\perp}, \mathcal{M})$, $X \in \mathcal{L}(\mathcal{M}^{\perp})$, X is an isosymmetry and $\sigma(T) \subseteq \mathbf{D}^{-}$. Since T is hyponormal and A is self-adjoint, $E = 0$. Therefore, X is hyponormal and so X is a contraction. The proof of Proposition 5.22 is complete by an application of Proposition 5.18.

We now classify isosymmetries T such that either $Im(T) \geq 0$ or $Im(T) \leq 0$.

Lemma 5.24 *Let $T \in \mathcal{L}(\mathcal{H})$. If $Im(T) \geq 0$ and T is an isosymmetry, then T lifts to a normal operator which is an isosymmetry.*

Proof. The proof of Lemma 5.24 will follow from a sequence of three reductions.

Reduction 1. *If Lemma 5.24 holds for all T such that T is an isosymmetry, $Im(T) \geq 0$ and $\sigma(T) \subseteq \mathbf{D}^{-}$, then Lemma 5.24 holds for all isosymmetries T such that $Im(T) \geq 0$.*

Suppose that T is an isosymmetry, $Im(T) \geq 0$ and $\sigma(T) \not\subseteq \mathbf{D}^{-}$. By Lemma 5.10, there exists an invariant subspace $\mathcal{M}$ for T such that T has the block operator form

$$T = \begin{bmatrix} A & E \\ 0 & X \end{bmatrix} \tag{5.25}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ where $A \in \mathcal{L}(\mathcal{M})$, A is self-adjoint, $E \in \mathcal{L}(\mathcal{M}^{\perp}, \mathcal{M})$, $X \in \mathcal{L}(\mathcal{M}^{\perp})$, $\sigma(X) \subseteq \mathbf{D}^{-}$ and X is an isosymmetry. Since $Im(T) \geq 0$ and A is self-adjoint, $E = 0$. Therefore, $Im(X) \geq 0$ and X is an isosymmetry. Thus, by the hypothesis of Reduction 1, X is subnormal.

This shows that $T = A \oplus X$ is subnormal and so by Proposition 3.29 the conclusion of Proposition 5.24 holds for T. This completes the proof of Reduction 1.

Reduction 2. *If Lemma 5.24 holds for all T such that T is an isosymmetry, $Im(T) \geq 0$, $\sigma(T) \subseteq \mathbf{D}^-$ and $0 \notin \sigma_p(Im(T))$, then Lemma 5.24 holds for all isosymmetries T such that $Im(T) \geq 0$.*

Suppose that T is an isosymmetry, $Im(T) \geq 0$ and $\sigma(T) \subseteq \mathbf{D}^-$. Let $\mathcal{M} = \ker(Im(T))$. If $\mathcal{M} = \{0\}$ or $\mathcal{M} = \mathcal{H}$, then Lemma 5.24 holds for T by the hypothesis of Reduction 2 or Lemma 4.7. Suppose that $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}$. Now, if $h \in \mathcal{M}$, then

$$\langle Im(T)Th, Th\rangle = \langle T^* Im(T)Th, h\rangle = \langle Im(T)h, h\rangle = 0.$$

Since $Im(T) \geq 0$, the above equation implies that $\mathcal{M}$ is invariant for T and so T has the block operator form

$$T = \begin{bmatrix} A & E \\ 0 & X \end{bmatrix} \tag{5.26}$$

with respect to the Hilbert space decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ where A is self-adjoint, $Im(X)$ is injective and X is an isosymmetry. Since $Im(T) \geq 0$ and A is self-adjoint, (5.26) implies that $E = 0$. Therefore, $Im(X) \geq 0$ and, by the hypothesis of Reduction 2, X is subnormal. This shows that $T = A \oplus X$ is subnormal and so, by Proposition 3.29, the conclusion of Lemma 5.24 holds for T. This completes the proof of Reduction 2.

Reduction 3. *Lemma 5.24 holds for all T.*

By Reduction 2 and Proposition 3.29, we need only show that if T is an isosymmetry, $\mathrm{Im}(T) \geq 0$, $\sigma(T) \subseteq \mathbf{D}^-$ and $0 \notin \sigma_p(Im(T))$, then T is subnormal.

First suppose that T is finitely cyclic. Let $\pi_0 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the natural map. Since $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a unital C^*-algebra, there exist a Hilbert space $\mathcal{M}$ and an injection unital *-representation $\pi : \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to \mathcal{L}(\mathcal{M})$. By Theorem 5.1, $\sigma_e(T) \subseteq \partial\mathbf{D} \cup [-1,1]$ and π_0 is normal. Thus, $\pi(\pi_0(T))$ is normal and $\sigma(\pi(\pi_0(T))) \subseteq K$. Therefore, $\pi(\pi_0(T))$ is a contraction and $\pi(\pi_0(1 - T^*T)) \geq 0$. Since π is injective, $\pi_0(1 - T^*T) \geq 0$ and so $1 - T^*T$ is essentially positive.

Let V be a partial isometry such that $T = V\,|T|$.

We now claim that $1 - T^*T$ does not have any negative eigenvalues. Suppose that $t > 0$, $h \in \mathcal{H}$ and $(1 - T^*T)h = -th$. Now, $Ph = \sqrt{1+t}\,h$ and so

$$tPV^*h = T^*(th) = T^*\Delta_T h = \Delta_T Th = (P^2 - 1)VPh = \sqrt{1+t}\,(P^2 - 1)Vh\,.$$

Therefore,

$$\begin{aligned} t\sqrt{1+t}\,\langle V^*h, h\rangle &= t\langle V^*h, \sqrt{1+t}\,h\rangle \\ &= t\langle V^*h, Ph\rangle \\ &= t\langle PV^*h, h\rangle \\ &= \sqrt{1+t}\langle (P^2 - 1)Vh, h\rangle \\ &= \sqrt{1+t}\langle Vh, (P^2 - 1)h\rangle \\ &= \sqrt{1+t}\langle Vh, th\rangle \\ &= t\sqrt{1+t}\langle Vh, h\rangle\,. \end{aligned}$$

Thus, $\langle Vh, h\rangle \in \mathbf{R}$ and

$$\langle Im(T)h, h\rangle = Im(\langle Th, h\rangle) = Im(\langle VPh, h\rangle) = tIm(\langle Vh, h\rangle) = 0\,.$$

Since $Im(T) \geq 0$, the above equation implies that $Im(T)h = 0$. Since we assumed that $0 \notin \sigma_p(Im(T))$, $h = 0$. In summary, $1 - T^*T$ does not have any negative eigenvalues.

Now, since $1 - T^*T$ is self-adjoint, essentially positive and has no negative eigenvalues, $1 - T^*T \geq 0$. By Proposition 5.18, the conclusion of Lemma 5.24 holds for T. This completes the proof of Reduction 3 and the proof of Lemma 5.24.

As a simple corollary of Lemma 5.24 and Lemma 4.9, we obtain the following.

Lemma 5.27 *Let $T \in \mathcal{L}(\mathcal{H})$. If $Im(T) \leq 0$ and T is an isosymmetry, then T lifts to a normal operator which is an isosymmetry.*

We now classify all isosymmetries T such that $T^*T \geq 1$.

Theorem 5.28 *Let T be an isosymmetry. If $T^*T \geq 1$, then T is (unitarily equivalent to)*

$$\begin{bmatrix} V & E \\ 0 & A \end{bmatrix} \Bigg| \mathcal{M} \tag{5.29}$$

where V is an isometry, A is self-adjoint, $\operatorname{ran}(E)^- = \ker(V^*)$, $E^*E + A^2 = \|T^*T\|$ and $\mathcal{M}$ is an invariant subspace for the above block operator.

Proof. If $\|T^*T\| \leq 1$, then T is a contraction. If T is an isosymmetry, T is a contraction and $T^*T \geq 1$, then by Proposition 5.18, T is subnormal. Since T is subnormal and $T^*T \geq 1$, T is an isometry and the conclusion of Theorem 5.28 holds. For the rest of the proof, we shall assume that $\|T^*T\| > 1$.

Recall that $\Delta_T = T^*T - 1$. Let $\sigma = \|\Delta_T\|^{\frac{1}{2}}$.

We will attach to the isosymmetry T a certain self-adjoint operator A and a certain contraction C. The extension of T will be constructed from A and an isometric dilation S of C^*.

Let $T \in \mathcal{L}(\mathcal{H})$, set

$$\delta = \left(1 - \frac{1}{\sigma^2}\,\Delta_T\right)^{\frac{1}{2}}$$

and let

$$\mathcal{H}_0 = (\operatorname{ran} \delta)^- .$$

Define $C_0 \colon (\operatorname{ran} \delta T) + (\mathcal{H}_0 \ominus (\operatorname{ran} \delta T)) \to \mathcal{H}_0$ by

$$C_0(\delta T x) = \delta x \ , \quad x \in \mathcal{H} ,$$

and

$$C_0 \,|\, \mathcal{H}_0 \ominus (\operatorname{ran} \delta T)^- = 0 .$$

To see that C_0 is well defined and extends by continuity to a contraction

$$C \colon \mathcal{H}_0 \to \mathcal{H}_0 ,$$

observe that

$$\begin{aligned} T^*\delta^2 T - \delta^2 &= (yx-1)(1-\frac{1}{\sigma^2}(yx-1))(T) \\ &= (yx-1)(1-\frac{1}{\sigma^2}(x^2-1))(T) \\ &= \frac{1}{\sigma^2}(yx-1)(\sigma^2+1-x^2)(T) \\ &= \frac{1}{\sigma^2}\Delta_T(\sigma^2+1-T^2) \\ &\geq 0 \qquad\quad \text{(Lemma 5.12)} \end{aligned}$$

and that if $h \in \mathcal{H}$ and $k \in \mathcal{H}_0 \ominus (\operatorname{ran} \delta T)^-$, then

$$\|\delta T h + k\|^2 - \|C_0(\delta T h + k)\|^2 = \frac{1}{\sigma^2}\langle \Delta_T(\sigma^2+1-T^2)h, h\rangle + \|k\|^2 . \tag{5.30}$$

Also note that (5.30) implies the defect identity,

$$(\delta T)^*(1 - C^*C)\delta T = \frac{1}{\sigma^2}\Delta_T(\sigma^2+1-T^2) . \tag{5.31}$$

In (5.31), δT is regarded as an operator into $\mathcal{H}_0$.

Set $\mathcal{H}_1 = (\operatorname{ran} \Delta_T^{\frac{1}{2}})^-$. Let $A_0 \colon \operatorname{ran} \Delta_T^{\frac{1}{2}} \to \operatorname{ran} \Delta_T^{\frac{1}{2}}$ be defined by

$$A_0(\Delta_T^{\frac{1}{2}} x) = \Delta_T^{\frac{1}{2}} T x \quad , \quad x \in \mathcal{H}.$$

By Lemma 4.27, $\ker \Delta_T$ is invariant for T and so A_0 is well-defined. Since $\Delta_T T$ is self-adjoint, A_0 is symmetric. We will prove that A_0 is bounded after defining E_0.

Let S^* be a coisometric extension of C acting on a space $\mathcal{K}_0$. That is, let $\mathcal{K}_0$ be a Hilbert space containing $\mathcal{H}_0$, $S \in \mathcal{L}(\mathcal{K}_0)$ and S be an isometry such that $\mathcal{H}_0$ is invariant for S^* and

$$C = S^* \,|\, \mathcal{H}_0 . \tag{5.32}$$

Define $E_0 \colon \operatorname{ran} \Delta_T^{\frac{1}{2}} \to (1 - SS^*)\mathcal{K}_0$ by setting

$$E_0 \Delta_T^{\frac{1}{2}} x = \sigma(1 - SS^*)\delta T x \ , \quad x \in \mathcal{H}.$$

Observe by using (5.31) and (5.32) that

$$\begin{aligned} \|A\Delta_T^{\frac{1}{2}} h\|^2 + \|E_0 \Delta_T^{\frac{1}{2}} h\|^2 &= \langle \Delta_T T^2 h, h\rangle + \sigma^2 \langle (1 - SS^*)\delta T h, \delta T h\rangle \\ &= \langle \Delta_T T^2 h, h\rangle + \sigma^2 \langle (1 - C^* C)\delta T h, \delta T h\rangle \\ &= \langle \Delta_T T^2 h, h\rangle + \langle \Delta_T (\sigma^2 + 1 - T^2) h, h\rangle \\ &= (\sigma^2 + 1)\|\Delta_T^{\frac{1}{2}} h\|^2 . \end{aligned}$$

Hence, E_0 is well-defined, bounded and extends by continuity to

$$E \colon \mathcal{K}_1 \to (1 - SS^*)\mathcal{K}_0 ,$$

A_0 is bounded and extends by continuity to a self-adjoint operator

$$A \colon \mathcal{H}_1 \to \mathcal{H}_1 .$$

Observe that

$$S^* E = 0, \tag{5.33}$$

$$E^*E + A^2 = \sigma^2 + 1 \tag{5.34}$$

and

$$\sigma(1 - SS^*)\delta T = E\Delta_T^{\frac{1}{2}}. \tag{5.35}$$

Now note that

$$B = \begin{bmatrix} S & E \\ 0 & A \end{bmatrix}$$

acting on $\mathcal{K}_0 \oplus \mathcal{K}_1$ is an isosymmetry, $\sigma(\Delta_T) = \{0, \sigma^2\}$ (by (5.33), (5.34), (5.35)), and that the map $L\colon \mathcal{H} \to \mathcal{K}_0 \oplus \mathcal{K}_1$ defined by

$$L(x) = \delta x \oplus \frac{1}{\sigma}\, \Delta_T^{\frac{1}{2}}\, x$$

is an isometry. Thus, the proof of Theorem 5.28 will be complete if it is shown that

$$LT = BL. \tag{5.36}$$

To see that (5.36) holds, note that the equation has the two components,

$$\delta Tx = S\delta x + \frac{1}{\sigma} E\Delta_T^{\frac{1}{2}} x, \tag{5.37}$$

and

$$\frac{1}{\sigma}\, \Delta_T^{\frac{1}{2}} Tx = A\frac{1}{\sigma}\, \Delta_T^{\frac{1}{2}}\, x, \tag{5.38}$$

for all $x \in \mathcal{H}$. To prove (5.37) note using (5.35), (5.32), and the definition of C, that

$$\begin{aligned} \delta Tx &= SS^*\delta Tx + (1 - SS^*)\delta Tx \\ &= SS^*\delta Tx + \frac{1}{\sigma} E\Delta_T^{\frac{1}{2}} x \\ &= SC\delta Tx + \frac{1}{\sigma} E\Delta_T^{\frac{1}{2}} x \\ &= S\delta x + \frac{1}{\sigma} E\Delta_T^{\frac{1}{2}} x. \end{aligned}$$

Note that (5.38) holds due to the definition of A. This concludes the proof of Theorem 5.28.

Bibliography

[Ag1] Agler, J., *Subjordan operators*, Thesis, Indiana University, 1980.

[Ag2] Agler, J., "An Abstract Approach To Model Theory",*Survey of Some recent results in operator theory, volume II*, Pitman Research notes in Mathematics Series,No. 192.

[Ag3] Agler, J., "A disconjugacy theorem for Toeplitz operators",*American Journal Of Mathematics*, **112**(1980),1-14.

[Ag4] Agler, J. "Hypercontractions and subnormality",*Journal Of Operator Theory*, **13**(1985),203-217.

[Ag5] Agler, J., "The Arveson extension theorem and coanalytic models," *Integral Equations and Operator Theory* **5** (1982), 608-631.

[Ag6] Agler, J.,"Subjordan operators: Bishop's theorem, spectral inclusion, and spectral sets", *Journal of Operator Theory*, **7**(1982),373-395.

[Ag7] Agler, J., " Rational Dilation of an annulus", *Annals of Mathematics*, **121**(1985), 537-563.

[Ag8] Agler, J., "Family Theory", preprint.

[Ag-S] Agler, J., Stankus, M. "m-isometric Linear Transformations on Hilbert Space", to appear.

[Arv] Arveson, W.B.,"Subalgebras of C* algebras", *Acta Math.* 123 (1969),141-224.

[Be] Beals, R., "topics in operator theory", Chicago Lecutres in Mathematics, The University Of Chicago Press And London, 1971.

[Ber] Berberian,S. K., *Approximate proper values*, Proc. Amer. Math.
Soc., 13(1962), 111-114.

[B-D-F] Brown L., Douglas, R. G. , and Fillmore, P., *Unitarily equivalence modulo compact operators and extensions of C^*-algebras*, Proc. Conf. Operator Theory, Lecture Notes in Math, vol. 345, Springer, Berlin, 1973, pp. 58-128.

[B-H] Helton, J.W., Ball, J.A., "Nonnormal dilations, disconjugacy, and constrained spectral factorization",*Integral Equations and Operator Theory*,bf 312(1980),216-309.

[Bir-Sol] Birman, M. S., and Solomjak, M. Z., *Spectral Theory of self-adjoint Operators in Hilbert Space*, D. Reidel Publishing Company, Dordrecht/Boston/Lancaster/Tokyo, 1986.

[C] Conway, J. B., *A Course in Functional Analysis*, New York, Berlin, Heidelberg, Tokyo, Springer-Verlag, 1985.

[C2] Conway, J. B., *Subnormal Operators*, Research Notes in Mathematics, Pitman Advanced Publishing Program, London, No. **51**, 1981.

[C-P] Curto, R.E. and Putinar, M., "Existence of non-subnormal polynomially hyponormal operators", B.A.M.S., **25**(1991) #2, 373-378.

[dB-R] de Branges, L., Rovnyak, J., *Square Summable Power Series*, New York, Holt, Rinehart and Winston, 1966.

[F] Fillmore, P. A., *Notes on operator theory*, New York, Van Nostrand, 1970.

[Fr] Franks, E., *Polynomially Subnormal Operator Tuples*, Thesis, Univ. of Calif. San Diego, 1991.

[H1] Helton, J. W., "Operators with a representation as multiplication by x on a Sobolev space," *Colloquia Math. Soc. Janos Bolyai* **5**, Hilbert Space Operators, Tihany, Hungary (1970), 279-287.

[H2] Helton, J. W., *Operator Theory, analytic functions, matrices, and electrical engineering*, Conference Board of the Mathematical Sciences by the American Mathematical Society, 1987.

[H3] Helton, J. W., "Infinite dimensional Jordan operators and Sturm-Liouville conjugate point theory," *Trans. Amer. Math. Soc.* **170** (1972), 305-331.

[H4] Helton, J. W., "Jordan Operators in infinite dimensions and Sturm-Liouville conjugate point theory," *Bulletin. Amer. Math. Soc.* **170** (1972), 305-331.

[Ha] Halmos, P. R., "A Hilbert Space Problem Book", Second Edition, Springer-Verlag, Te3xts in Graduate Mathematics, vol 19, 1982.

[H-W] Helton, J.W. and Wavrik,J. J., "Rules for Computer Simplification of the formulas in operator model theory and Linear Systems", in preparation.

[M1] McCullough, S. A., *3-isometries*, Thesis, Univ. Of. Calif. San Diego, 1987.

[M2] McCullough, S. A. , "SubBrownian Operators",*Journal of Operator Theory*, **22** (1989)#**2**,291-305.

[N] Nielsen, O.A., "Direct Integral Theory," *Lecture notes in Pure and Applied Mathematics*, vol. **61**, Marcel Dekker, Inc, New York and Basel, 1980.

[P] Pearcy, C.M., *Some recent developments in Operator Theory*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. **36**.

[R1] Richter, S., "Invariant subspaces of the Dirichlet shift," *J. reine agnew. Math.* **386** (1988), 205-220.

[R2] Richter, S., "A representation theorem for cyclic analytic two-isometries," *Trans. Amer. Math. Soc.*, to appear.

[R-R] Radjavi, H., Rosenthal, P., "On Roots of Normal Operators", *Jour. Of Math. Analysis and Appl.* **34**(1971),653-664.

[R-R2] Radjavi, H., Rosenthal, P., "Invariant Subspaces", Springer-Verlag, New York, 1973.

[SzN-F] Sz.-Nagy.j Foias, C., *Harmonic analysis of operators in Hilbert space*, North Holland, Amsterdam, 1970.